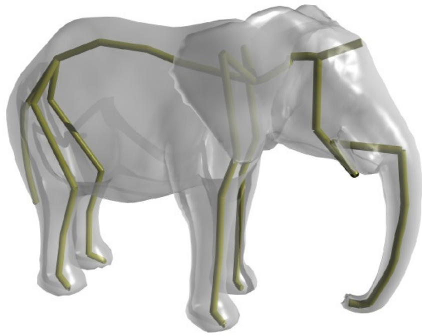
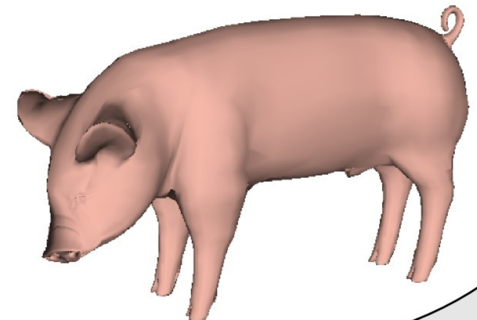


Geometric Surface Deformation



Goal of surface deformation

We want to modify the coordinates

No modification of the connectivity

Topological changes = complex

```
OFF
40 95 75
-0.175114 -0.047799 -0.046492
-0.199566 0.730914 -0.064795
-0.010689 0.674496 0.008900
-0.015538 0.153071 0.107408
-0.070148 0.767894 -0.116107
-0.053836 -0.702815 0.109714
-0.162416 -0.785481 0.088014
-0.112365 -0.782492 0.135482
-0.240928 0.031451 0.031966
-0.259289 0.200557 0.035420
0.296891 -0.707385 0.143375
-0.190129 -0.069002 0.109358
-0.010148 0.024179 -0.067283
-0.112968 -0.089127 0.092391
-0.185828 0.377372 -0.111155
3 20 4 1
3 34 11 13
3 12 30 0
3 30 13 17
3 23 22 21
3 29 38 17
3 32 0 13
3 14 0 37
3 24 4 21
3 14 32 1
3 24 2 22
3 3 12 25
3 4 24 15
3 21 15 26
3 35 34 13
3 19 32 13
3 19 13 27
```

Physically based approach ?

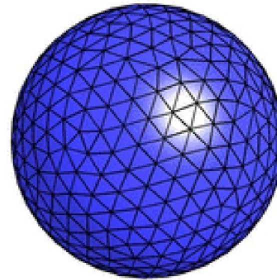
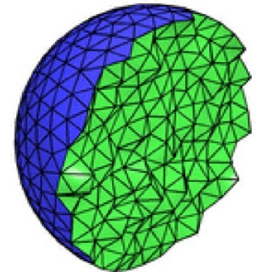
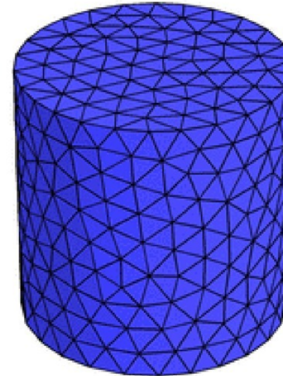
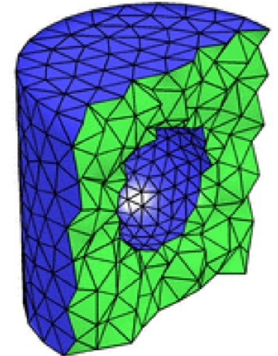
Express physical equations on volume elements
= PDE on tetrahedrons

Apply initial conditions (forces, speed, parameters) on the volumetric mesh

Solve the equations using numerical approaches:
Inversion of large sparse matrices

We wait ...

Iterate with different parameters

Non physically based approaches

+ Deform as wanted the geometry
+ Goal oriented
+ Only deal with what is seen
- Potentially introduce non realistic deformations
  => geometrical constraints

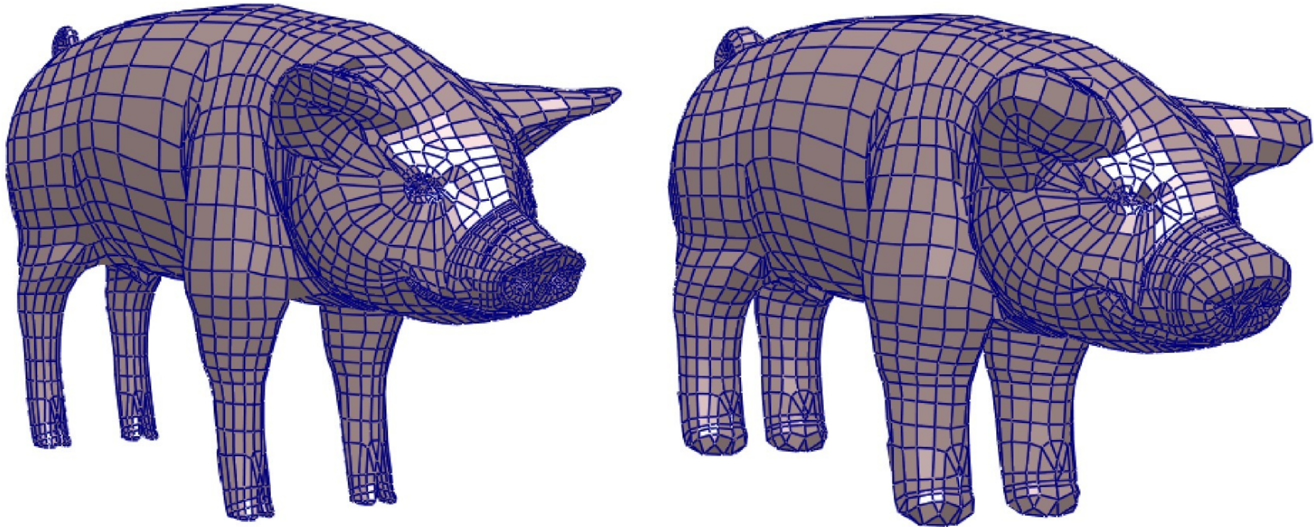
Mesh deformation

Surface animation / deformation

= find f such that

$$(\mathbf{y}_0, \mathbf{y}_1, \ldots, \mathbf{y}_n) = f(\mathbf{x}_0, \mathbf{x}_1, \ldots, \mathbf{x}_n)$$

We draw the mesh formed by $(\mathbf{y}_0, \mathbf{y}_1, \ldots, \mathbf{y}_n)$ with the original connectivity



How to choose f ?

Deformation functions

Few functions are known

$$f : \mathbf{x} \in \mathbb{R}^3 \mapsto \mathbf{y} \in \mathbb{R}^3$$

We take f as a linear/affine map

$$f = \mathrm{M}$$

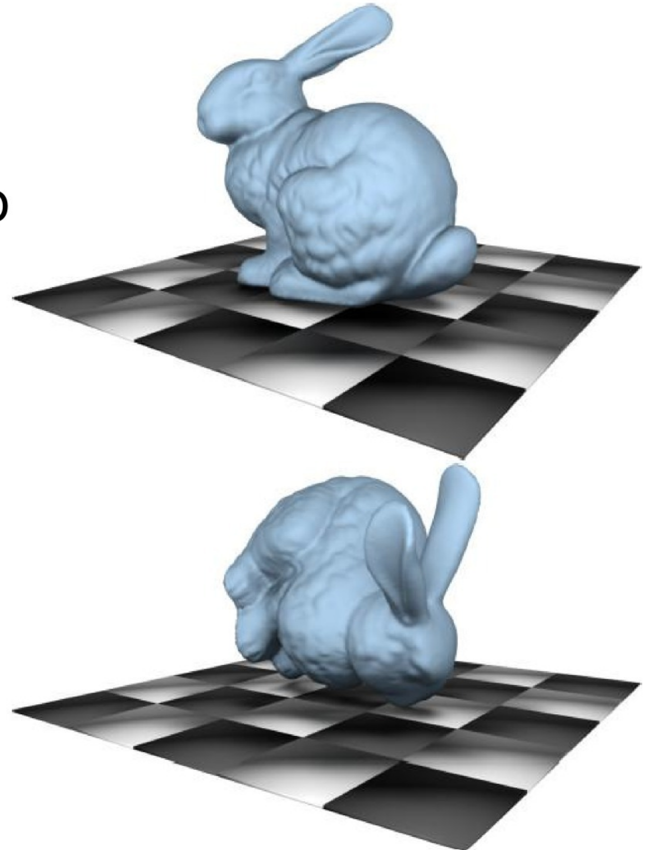
Isometry

Translations
Rotations
Symetries

$$\mathbf{y} = \mathrm{M}\,\mathbf{x} \;\; \& \;\; |\det(\mathrm{M})| = 1$$

Scaling

$$\mathbf{y} = \mathrm{diag}(s_1, s_2, s_3)\;\mathbf{x}$$

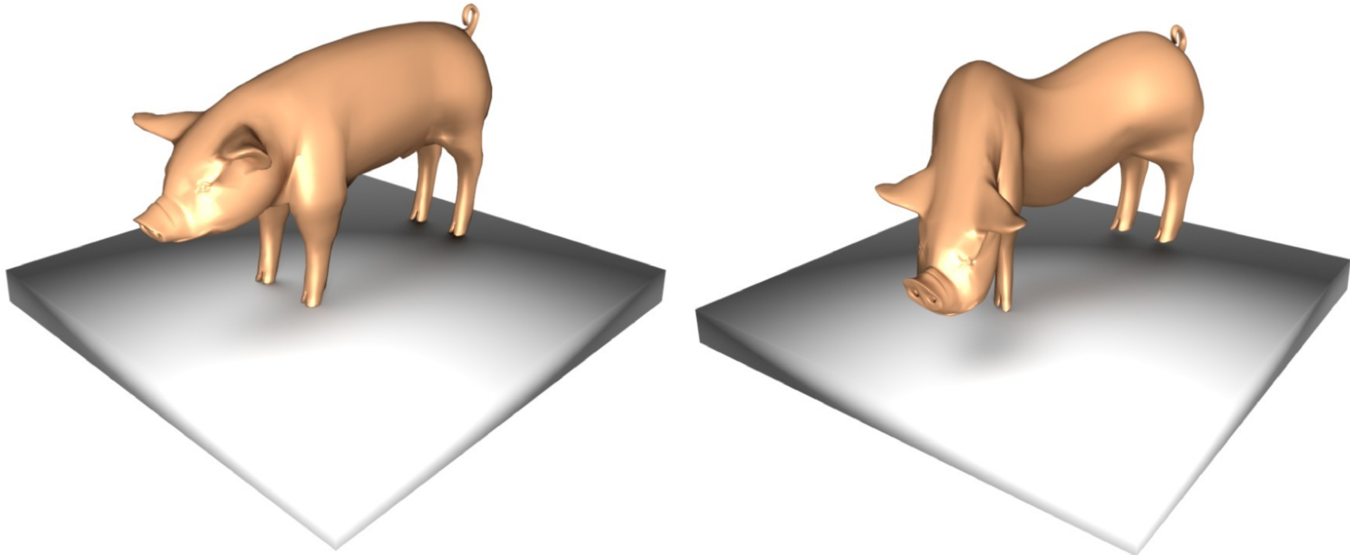


Rigid deformations

The parameter of a rigid transformation can vary in space

$$f : \mathbf{p} \in \mathbb{R}^3 \mapsto \mathbf{y} = \mathrm{M}(\mathbf{p})\ \mathbf{p}$$

ex. Translation depending of position



Rigid deformations

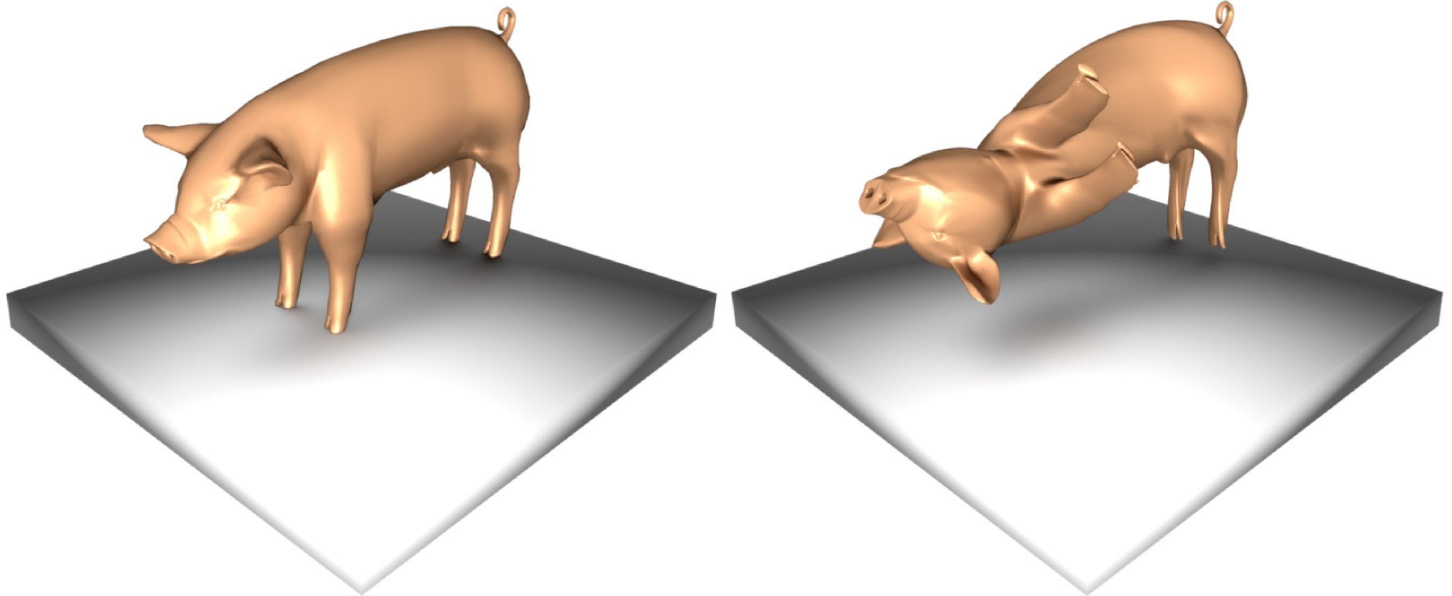
ex. Varying angle of rotation

$$\forall i,\ \mathbf{y}_i = \mathrm{R}\left(\mathbf{n}, \frac{z}{z_{\max}}\right)\mathbf{x}_i$$

```
for(k=0;k<N;k++)
{
  Vec x = mesh.get_vertex(k);
  double angle = x[0] * PI * time;
  Matrix R = Matrix::rotation(Vec(1,0,0),angle);
  mesh.set_vertex(k,R*x);
}
```

Rigid deformations

ex. Varying angle of rotation



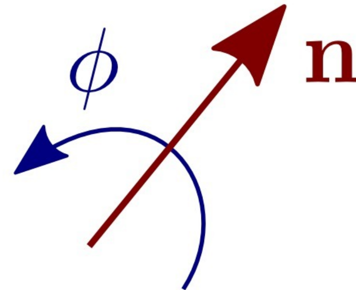
Rotation matrix

Every rotation can be parameterized by an **axe** and an **angle**

$$R(\mathbf{n},\phi) = \begin{pmatrix} \cos(\phi) + n_x^2(1-\cos(\phi)) & n_x n_y(1-\cos(\phi)) - n_z\sin(\phi) & n_y\sin(\phi) + n_x n_z(1-\cos(\phi)) \\ n_z\sin(\phi) + n_x n_y(1-\cos(\phi)) & \cos(\phi) + n_y^2(1-\cos(\phi)) & -n_x\sin(\phi) + n_y n_z(1-\cos(\phi)) \\ -n_y\sin(\phi) + n_x n_z(1-\cos(\phi)) & n_x\sin(\phi) + n_y n_z(1-\cos(\phi)) & \cos(\phi) + n_z n_z(1-\cos(\phi)) \end{pmatrix}$$

To apply a rotation:

$$\forall i,\ \mathbf{y}_i = \mathrm{R}\,\mathbf{x}_i$$



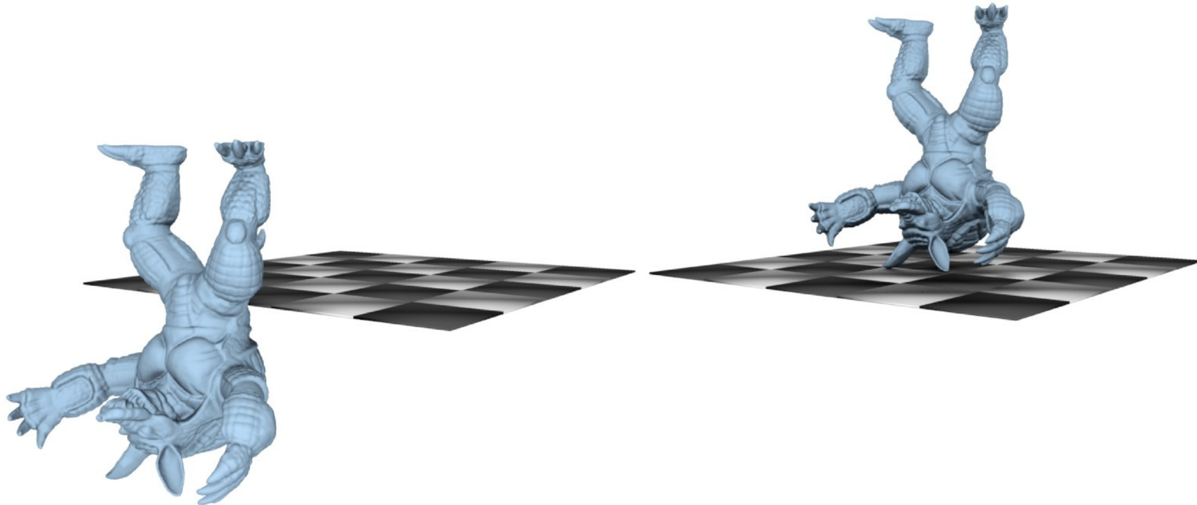
Rotation matrix

Should not forget the rotation center

$$\mathbf{q} = \mathrm{R}(\mathbf{p} - \mathbf{p}_0) + \mathbf{p}_0$$

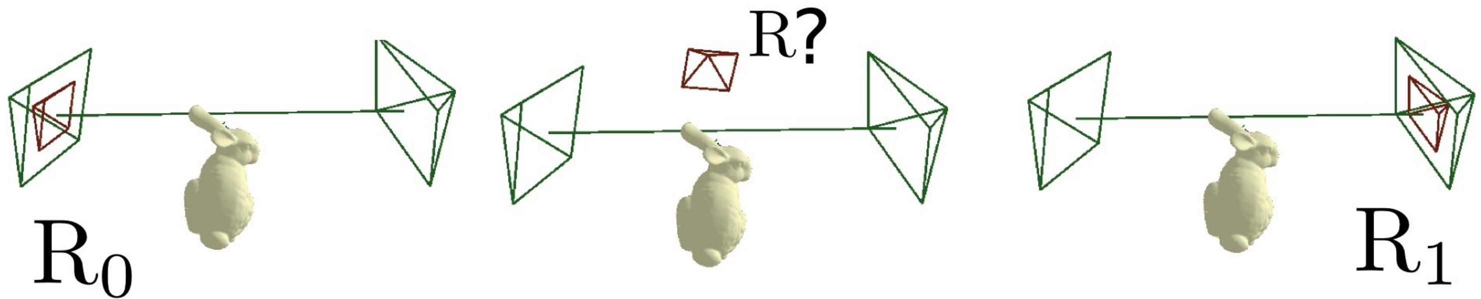
Or in its matrix form

$$\mathbf{q} = \mathrm{T}^{-1}\,\mathrm{R}\,\mathrm{T}\,\mathbf{p}$$



Rotation interpolation

How to interpolate rotation ?



Linear interpolation does not work

$$M = \alpha\,R_1 + (1-\alpha)\,R_2 \notin SO(3)$$

Problem: Rotation space is not a vectorial space

Rotation space is a manifold called Lie Group
(unit sphere in 4D)

Rotation interpolation

Generalization of linear interpolation in Lie Group

$$\mathrm{R} = \left(\mathrm{R}_2\,\mathrm{R}_1^{-1}\right)^{\alpha}\mathrm{R}_1$$

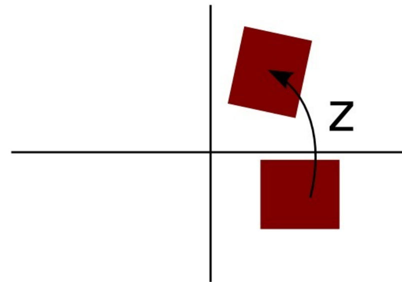
But: requires matrix exponential

- Computational cost
- Numerical instabilities

Quaternion

Remember that complex numbers can model rotation in 2D

$$z = e^{\mathbf{i}\theta}$$



$$z = x + y\,\mathbf{i}$$

$$|z| = 1$$

Quaternions are generalization of complex numbers in 3D

$$q = x + y\,\mathbf{i} + z\,\mathbf{j} + w\,\mathbf{k} \;\in \mathbb{R}^4$$

$$|q| = 1$$

Quaternion

Unit quaternion are rotations in 3D

$$q = x + y\,\mathbf{i} + z\,\mathbf{j} + w\,\mathbf{k}$$

$$|q| = 1$$

+ Short representation of rotation $q_1\, q_2 \Leftrightarrow \mathrm{R}_1 \mathrm{R}_2$

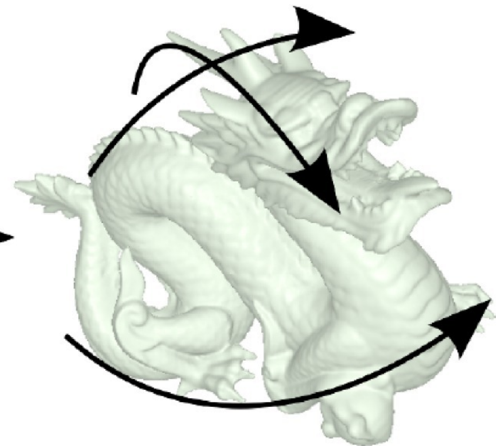
(non comutative product)

Algebre

rotations

$\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix}$

Lie
Group
SO(3)



Equivalence rotation/quaternion

Quaternion $q = (q_0, q_1, q_2, q_3) \;\Rightarrow\;$ Matrix R

$$\mathrm{R} = \begin{pmatrix} 1-2(q_1^2+q_2^2) & 2(q_0q_1-q_3q_2) & 2(q_0q_2+q_3q_1) \\ 2(q_0q_1+q_3q_2) & 1-2(q_0^2+q_2^2) & 2(q_1q_2-q_3q_0) \\ 2(q_0q_2-q_3q_1) & 2(q_1q_2+q_3q_0) & 1-2(q_0^2+q_1^2) \end{pmatrix}$$

Efficient representation of multiplication: $q = (\mathbf{v}, w)$

$$q_1\, q_2 = (w_0\mathbf{v}_1 + w_1\mathbf{v}_0 - \mathbf{v}_0 \times \mathbf{v}_1, w_0\, w_1 - \mathbf{v}_0 \cdot \mathbf{v}_1)$$

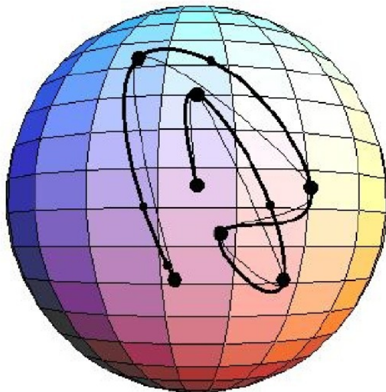
SLERP

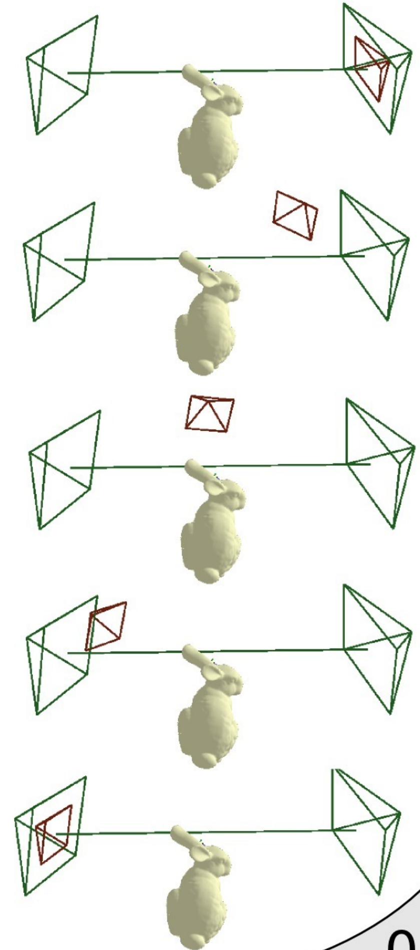
Quaternion interpolation => Rotation interpolation
SLERP: **S**pherical **L**inear Int**erp**olation

$$\text{SLERP}(q_0, q_1, \alpha) = \frac{\sin(\alpha(1-\theta))}{\sin(\theta)} q_0 + \frac{\sin(\alpha\theta)}{\sin(\theta)} q_1$$

$$\cos(\theta) = q_0 \cdot q_1$$

$$\alpha \in [0, 1]$$





Rigid transformations

- Non local transformation
- Complex to manipulate if axe/angle vary in space

=> We use mainly constant rigid transformation
=> Make them artificially piecewise local

$$\forall i,\ \mathbf{y}_i = \mathrm{M}(\varepsilon(i))\,\mathbf{x}$$

$\mathrm{M}(\varepsilon(i)) = \mathrm{I}$: for all undeformed set

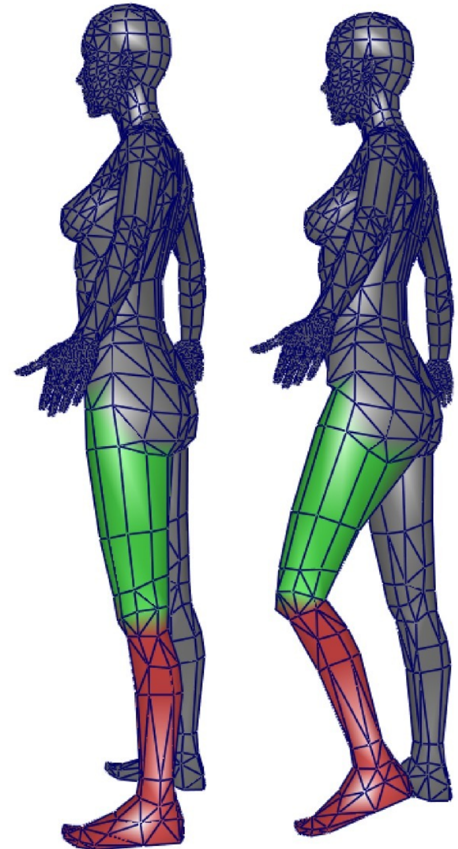
$\varepsilon(i)$: mesh segmentation

Local transformation

Define the regions ε

Apply a different transformation $\mathrm{M}(\varepsilon)$ on each region

Mainly use rotation R_ε

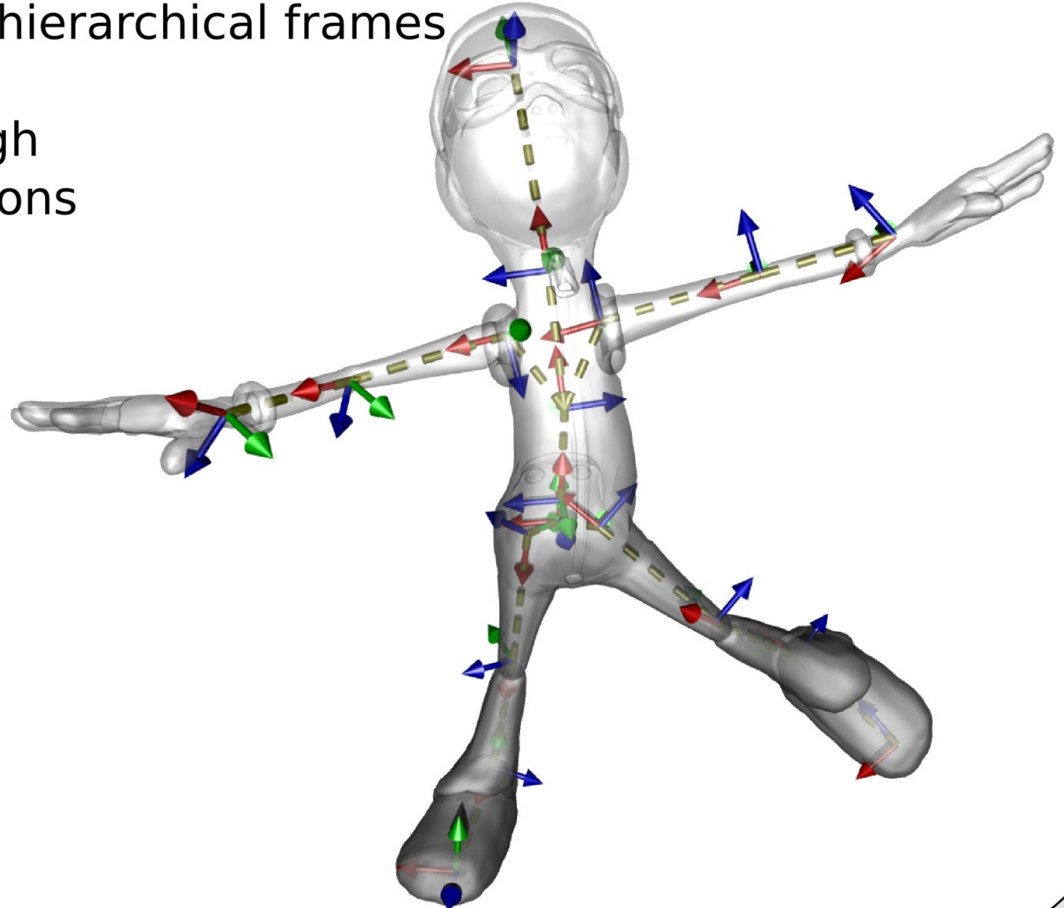


How to define the parameters ?

Animation skeleton

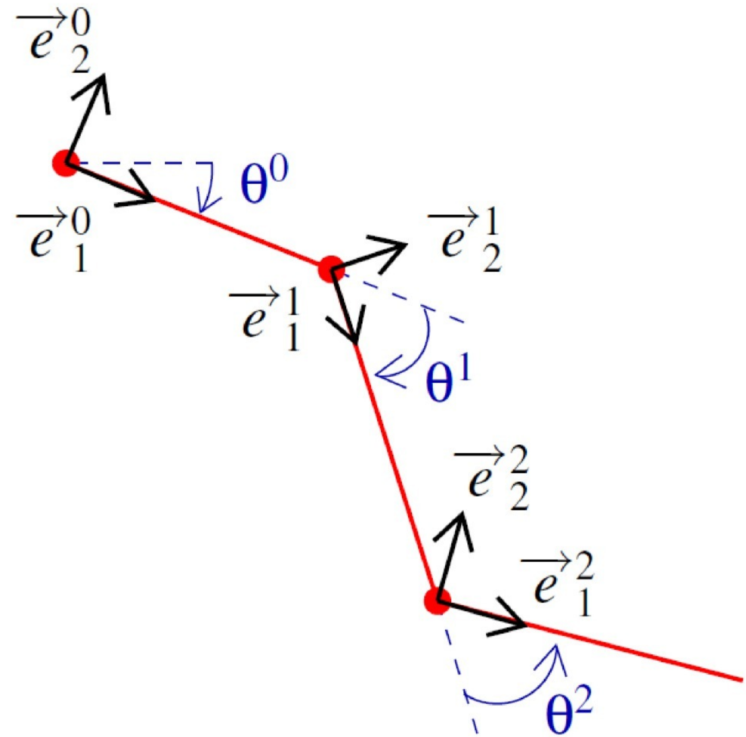
Object articulated using a skeleton
Skeleton=Set of hierarchical frames

Animation through
successive rotations



Animation skeleton

Hierarchical transformations

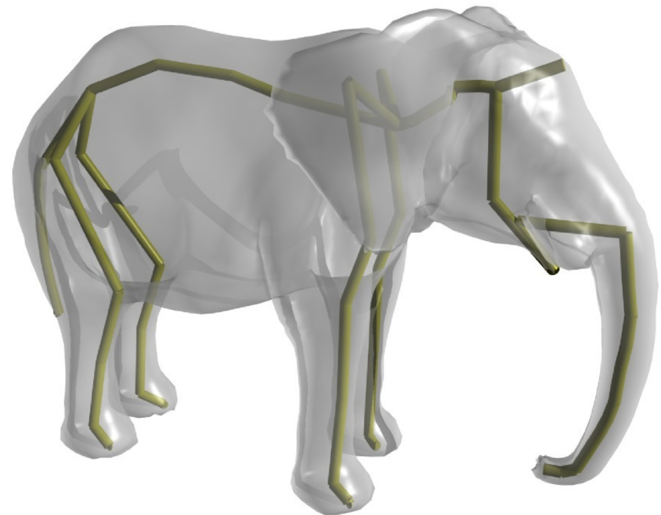


$$
\begin{aligned}
T_0 &= R_0 \\
T_1 &= T_0 M_1 R_1 M_1^{-1} = R_0 M_1 R_1 M_1^{-1} \\
T_2 &= T_1 M_2 R_2 T_2^{-1} = \dots \\
&\vdots \\
T_i &= \prod_{k=0}^{i} M_k R_k M_k^{-1}
\end{aligned}
$$

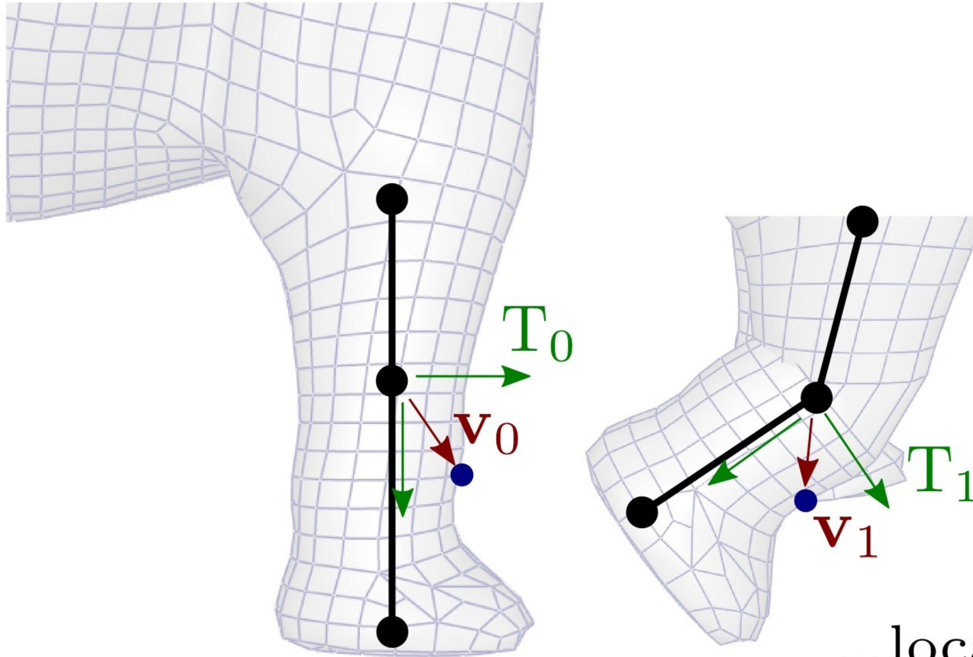
Rigid skinning

Link a set of vertices to the frame of a skeleton

Apply the frame deformation to all the vertices of the set



Rigid skinning



$$\underbrace{\mathbf{v}_0^{\text{local}}}_{T_0^{-1}\mathbf{v}_0} = \underbrace{\mathbf{v}_1^{\text{local}}}_{T_1^{-1}\mathbf{v}_1}$$

$$\boxed{\mathbf{v}_1 = T_1\, T_0^{-1}\mathbf{v}_0}$$

Skeleton

```
class Frame
{
  Matrix R;
  Matrix Bind;
  Joint *father;
  std::list <Joint*> son;
}
```

```
for(k_vertex=0;k_vertex<N_vertex;k_vertex++)
{
  int bone_dependency =
  skinning.bone_dependency(k_vertex);
  Matrix T = skeleton.get_matrix(bone_dependency);
  Matrix B =
  (skeleton.get_bind_pose(bone_dependency)).invert();
  deformed_mesh(k_vertex) = T*B * mesh(k_vertex);
}
```

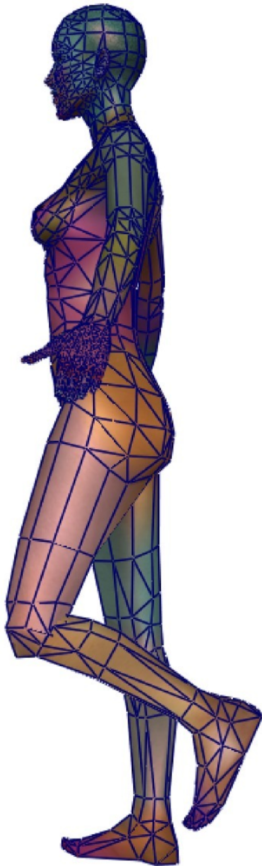
Good point of Skinning

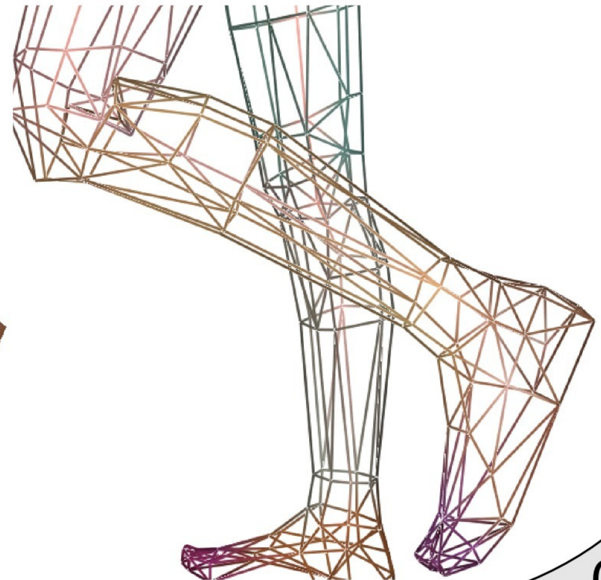
The skeleton is

+ Easy to construct
+ Easy to animate
+ Intuitif to manipulate an articulated character

Disadvantage of skinning

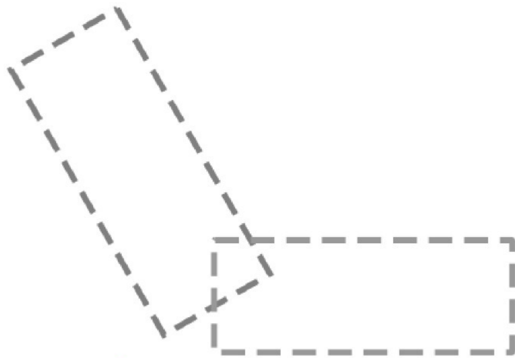
- Discontinuities

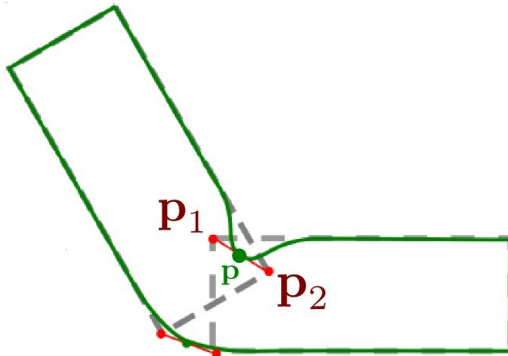




Interpolation of transformation

Idea: *Should interpolate deformation between frames*





$$\mathbf{p} = 0.5\,\mathbf{p}_1 + 0.5\,\mathbf{p}_2$$

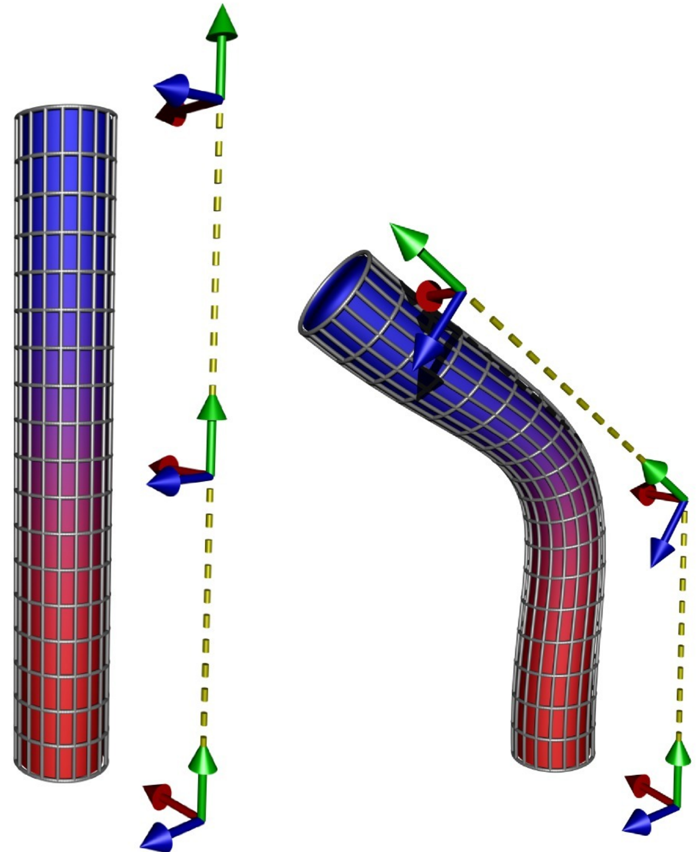
$$\mathbf{p} = 0.5\,\mathrm{T}_1\mathbf{p}_0 + 0.5\,\mathrm{T}_2\mathbf{p}_0$$

$$\mathbf{p} = \underbrace{\big(0.5\,\mathrm{T}_1 + 0.5\,\mathrm{T}_2\big)}_{\alpha\mathrm{T}_1 + (1-\alpha)\mathrm{T}_2}\,\mathbf{p}_0$$

Interpolation of transformation

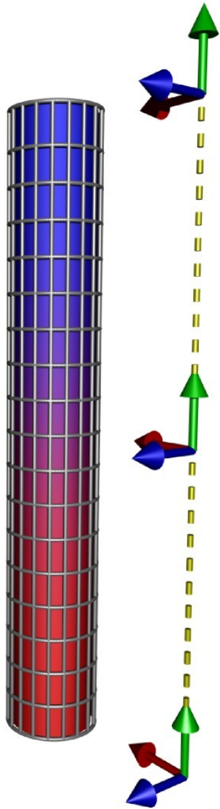
$$\mathrm{T} = \alpha \mathrm{T}_1 + (1 - \alpha)\mathrm{T}_2$$

How to compute the α

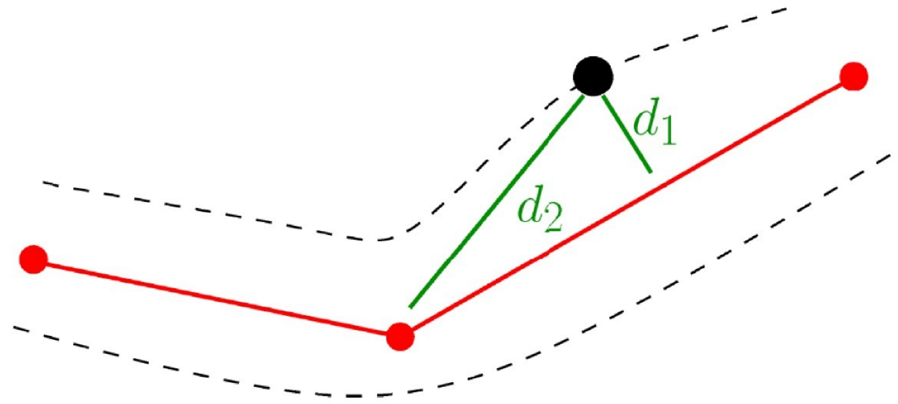


Skinning weights

Use of a distance function



$$\alpha = \frac{d_1^{-1}}{d_1^{-1} + d_2^{-1}}$$



Cylindrical distance

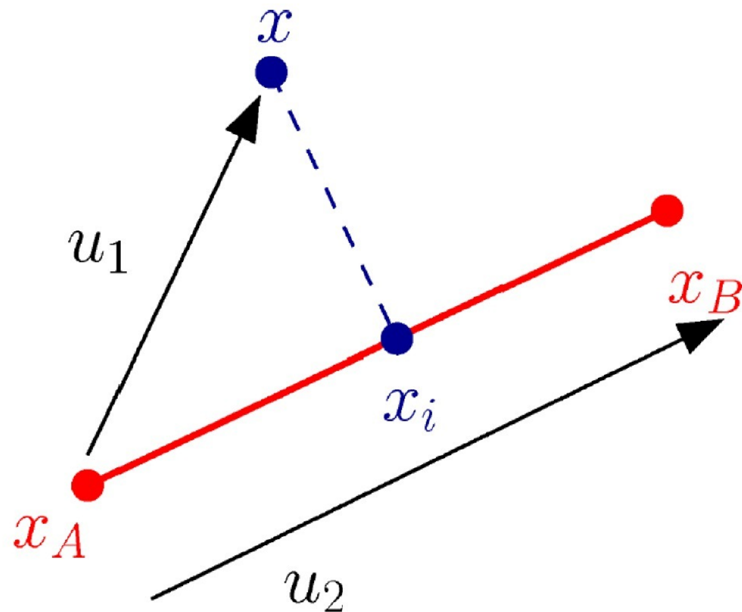
Shortest distance of a position **x** to a segment [AB]

$$\mathbf{u}_1 = \mathbf{x} - \mathbf{x}_A$$
$$\mathbf{u}_2 = \mathbf{x}_B - \mathbf{x}_A$$

$$\rho = \frac{\mathbf{u}_1 \cdot \mathbf{u}_2}{\|\mathbf{u}_2\|^2}$$

If $\rho < 0,\ \mathbf{x}_i = \mathbf{x}_A$
If $\rho > 1,\ \mathbf{x}_i = \mathbf{x}_B$
Else $\mathbf{x}_i = \mathbf{x}_A + \rho\,\mathbf{u}_2$

$$d = \|\mathbf{x} - \mathbf{x}_i\|$$



Data structure: skinning weights

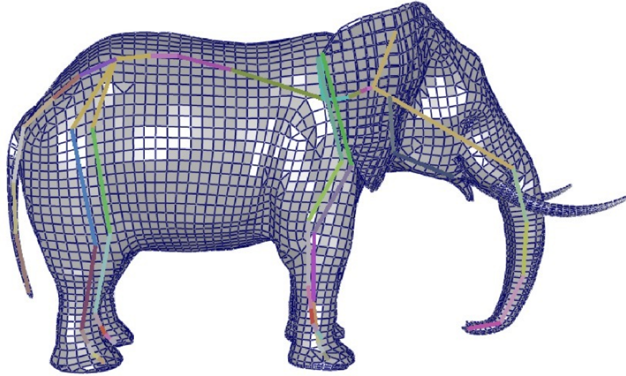
For every vertex i, we associate the pair (bone,weight)

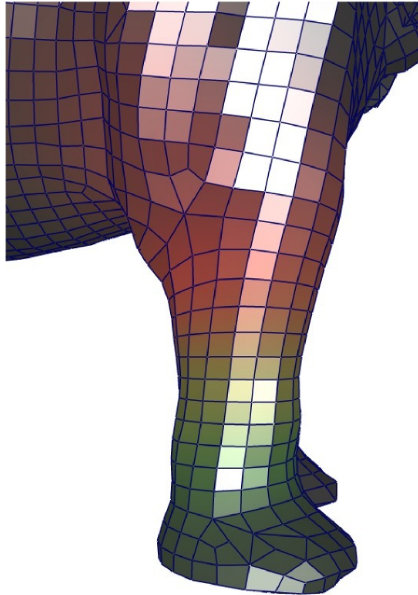
```
std::vector <std::vector <int>> bone_dependencies;
std::vector <std::vector <double>> skinning_weights;
for(k_vertex=0;k_vertex<N_vertex;k_vertex++)
{
  N_dep = bone_dependencies[k_vertex].size();
  for(k_dep=0;k_dep<N_dep;k_dep++)
  {
    bone = bone_dependencies[k_vertex][k_dep];
    weight = skinning_weight[k_vertex][k_dep];
  }
}
```

Skinning weights computation

```
//skinning weights
for(k_vertex=0;k_vertex<N_vertex;k_vertex++)
{
for(k_bone=0;k_bone<N_bone;k_bone++)
{
alpha = push_back(1/distance(k_vertex,k_bone));
if(alpha>epsilon)
{
skinning_weights[k_vertex].push_back(alpha);
bone_dependencies[k_vertex].push_back(k_bone);
}
}
norm_skinning_weights();
}
```

Skinning weights example





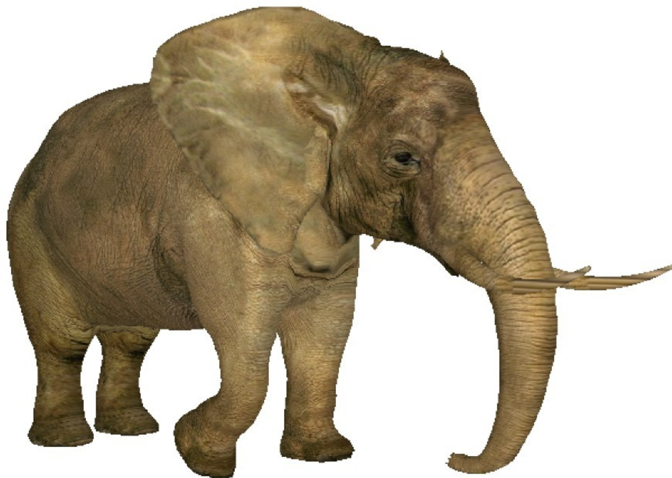
Smooth skinning

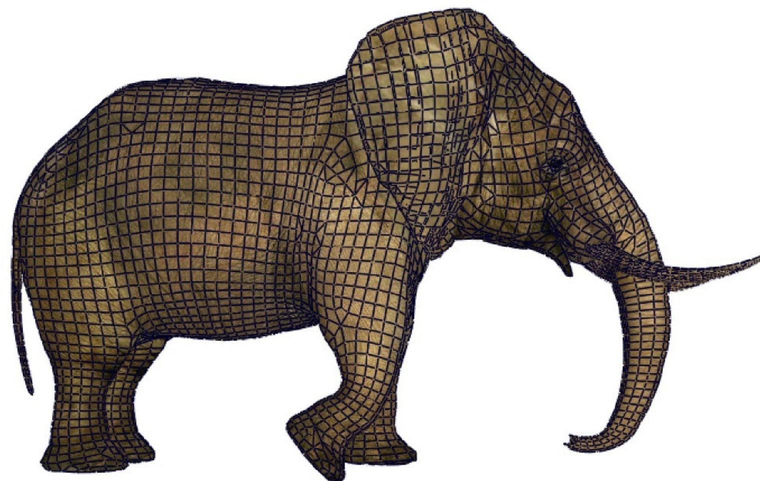
$$\mathbf{p} = \left(\sum_i \omega_i \, \mathrm{M}_i \right) \mathbf{p}_0$$

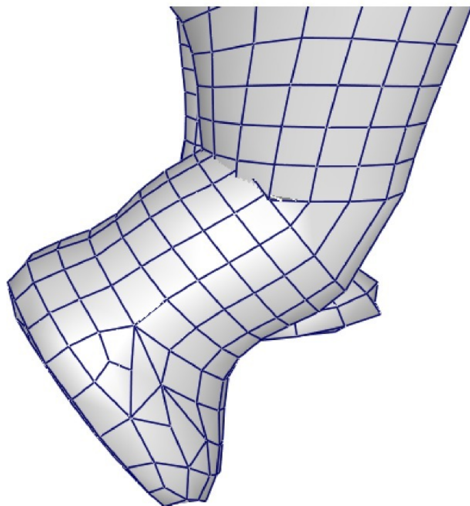
$$\sum_i \omega_i = 1$$

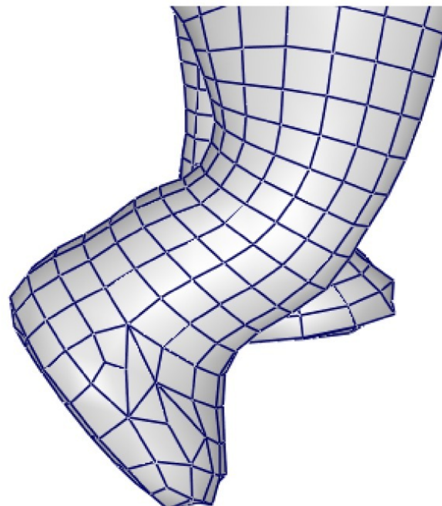
```
for(k_vertex=0;k_vertex<N_vertex;k_vertex++)
{ int N_dep = bone_dependencies[k_vertex].size();
  Matrix D;
  for(k_dep=0;k_dep<N_dep;k_dep++)
  { int k_bone = bone_dependency[k_vertex][k_dep];
    double weight = skinning_weights[k_vertex][k_dep];
    D += weight*skeleton.matrix(k_bone);}
    deformed_mesh(k_vertex) = D * mesh(k_vertex);
}
```

Smooth skinning examples



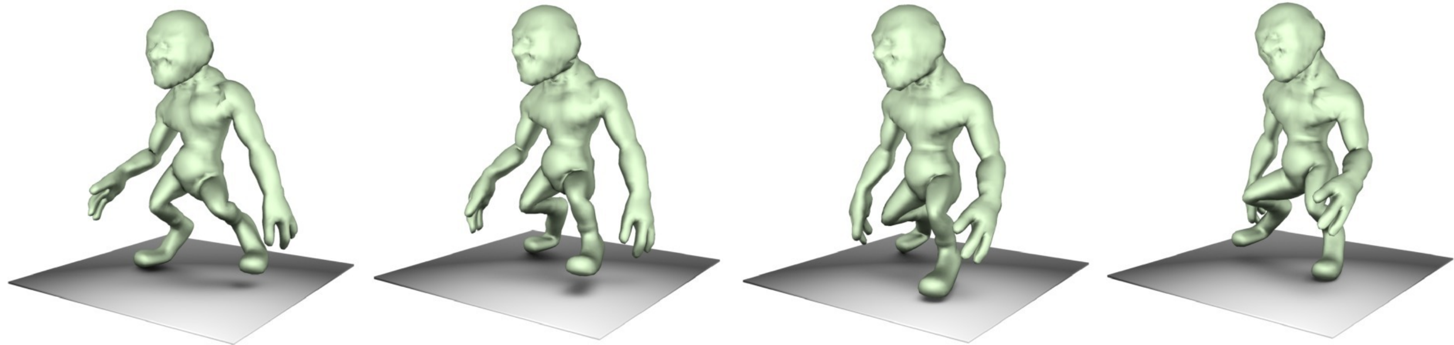






Animation

Angles varies through time

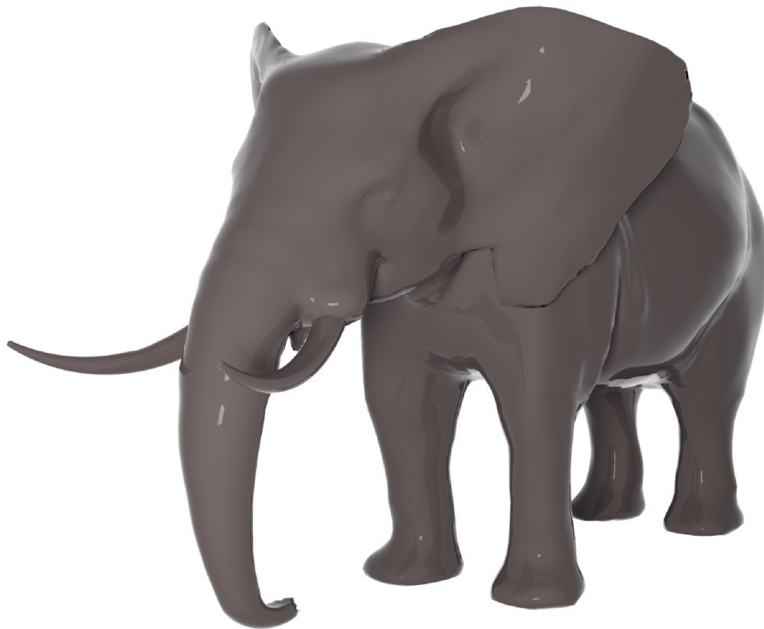


```
– <animation id="ele_R_elbow_rotateY">
  – <source id="ele_R_elbow_rotateY_ele_R_elbow_rotateY_ANGLE-input">
    – <float_array id="ele_R_elbow_rotateY_ele_R_elbow_rotateY_ANGLE-input-array" count="134">
        0.040000 0.080000 0.120000 0.160000 0.200000 0.240000 0.280000 0.320000 0.360000 0.400000 0.440000 0.480000 0.520000
        0.560000 0.600000 0.640000 0.680000 0.720000 0.760000 0.800000 0.840000 0.880000 0.920000 0.960000 1.000000 1.040000
        1.080000 1.120000
      </float_array>
    </source>
  – <source id="ele_R_elbow_rotateY_ele_R_elbow_rotateY_ANGLE-output">
    – <float_array id="ele_R_elbow_rotateY_ele_R_elbow_rotateY_ANGLE-output-array" count="134">
        0 -0.072670 -0.223444 -0.389830 -0.549435 -0.692583 -0.812962 -0.905247 -0.964565 -0.985765 -0.985721 -0.985668 -0.985607
        -0.985540 -0.961561 -0.887361 -0.755854 -0.560878 -0.285142 -0.201935 -0.173549 -0.175789 -0.182313 -0.192826 -0.207032
        -0.224636
      </float_array>
```

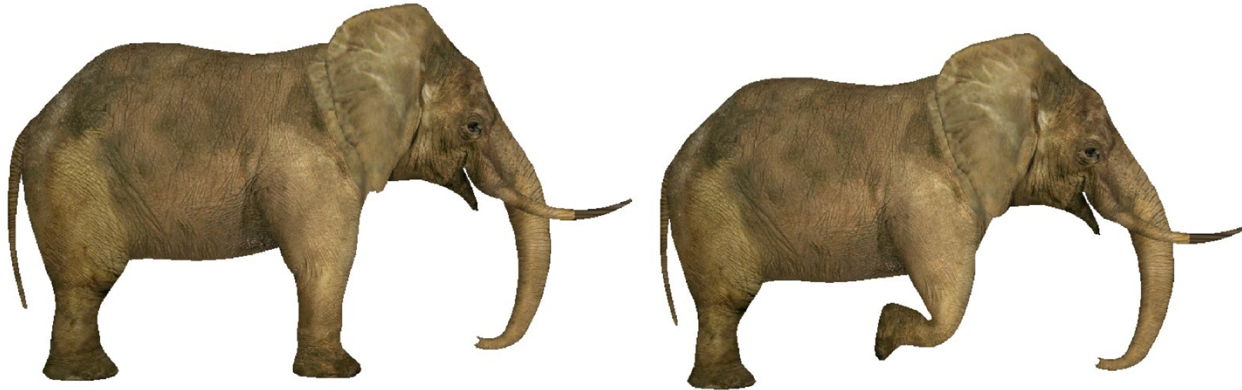
Smooth skinning, conclusion

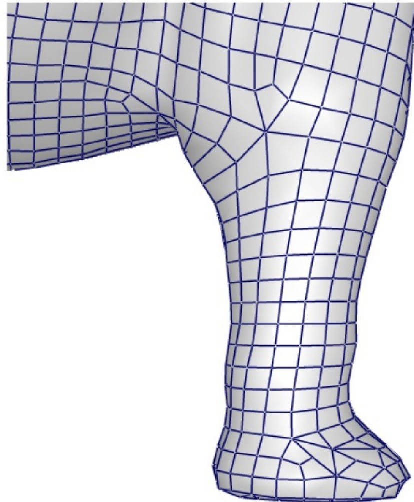
+ Fast computation
+ Intuitive skeleton manipulation

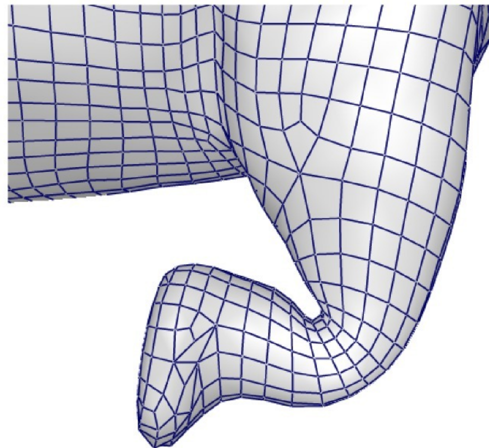
- Artifacts for large angles



Skinning artifact: Collapsing Elbow



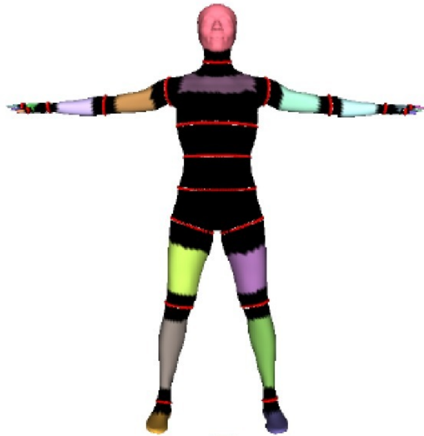


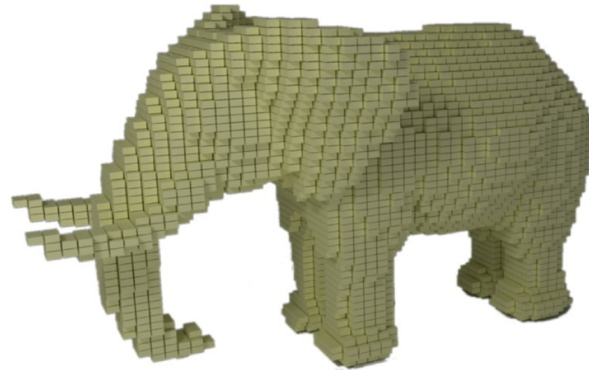


Skinning weight computation

Problem with cartesian distance

- Artistic tools to paint weights
- Use of geodesic distance



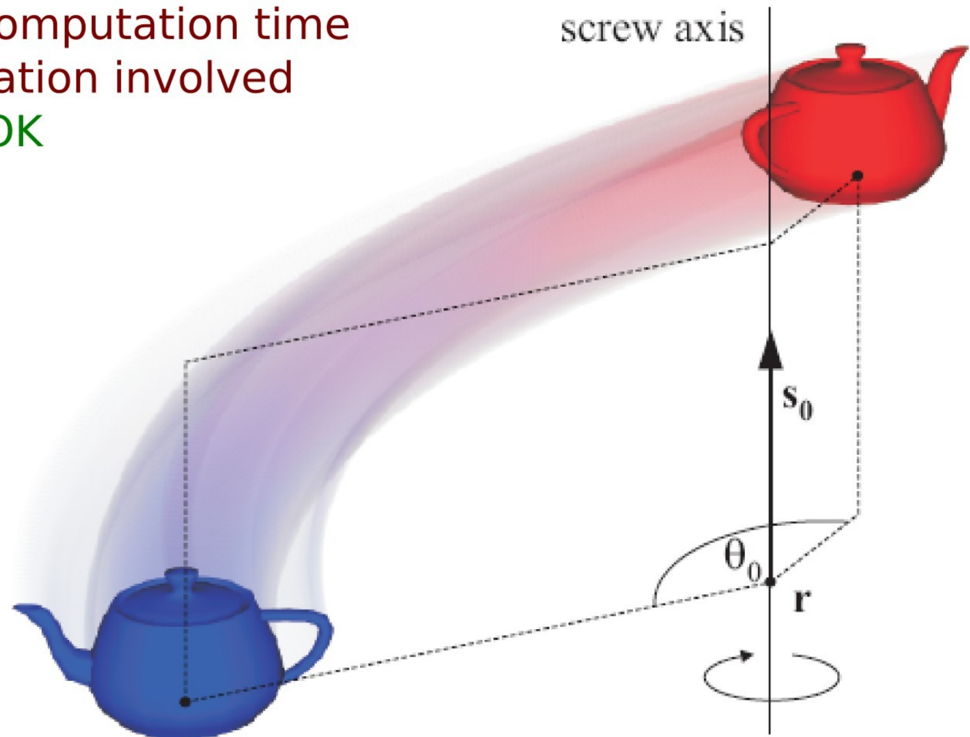


Skinning improvement

Improving interpolation

- Problem of linear interpolation of deformation

- Exponential map: computation time
- Quaternions: translation involved
- Dual Quaternions: OK



Other approaches

Skinning = **S**keleton **S**ubspace **D**eformation

Face animation: Shape interpolation
Smooth deformation: FFD
Hierachical deformation: Multires
Free deformation: Matrix inversion+constraints