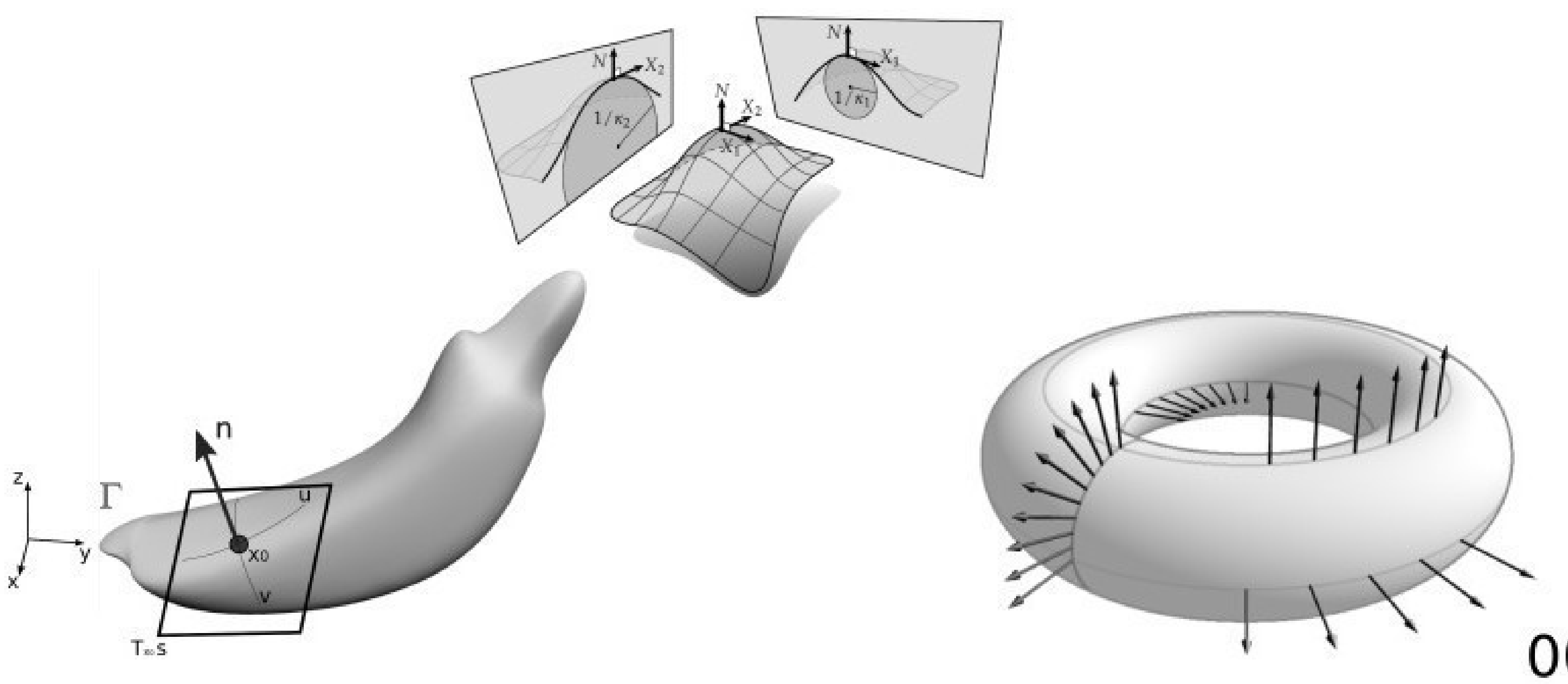
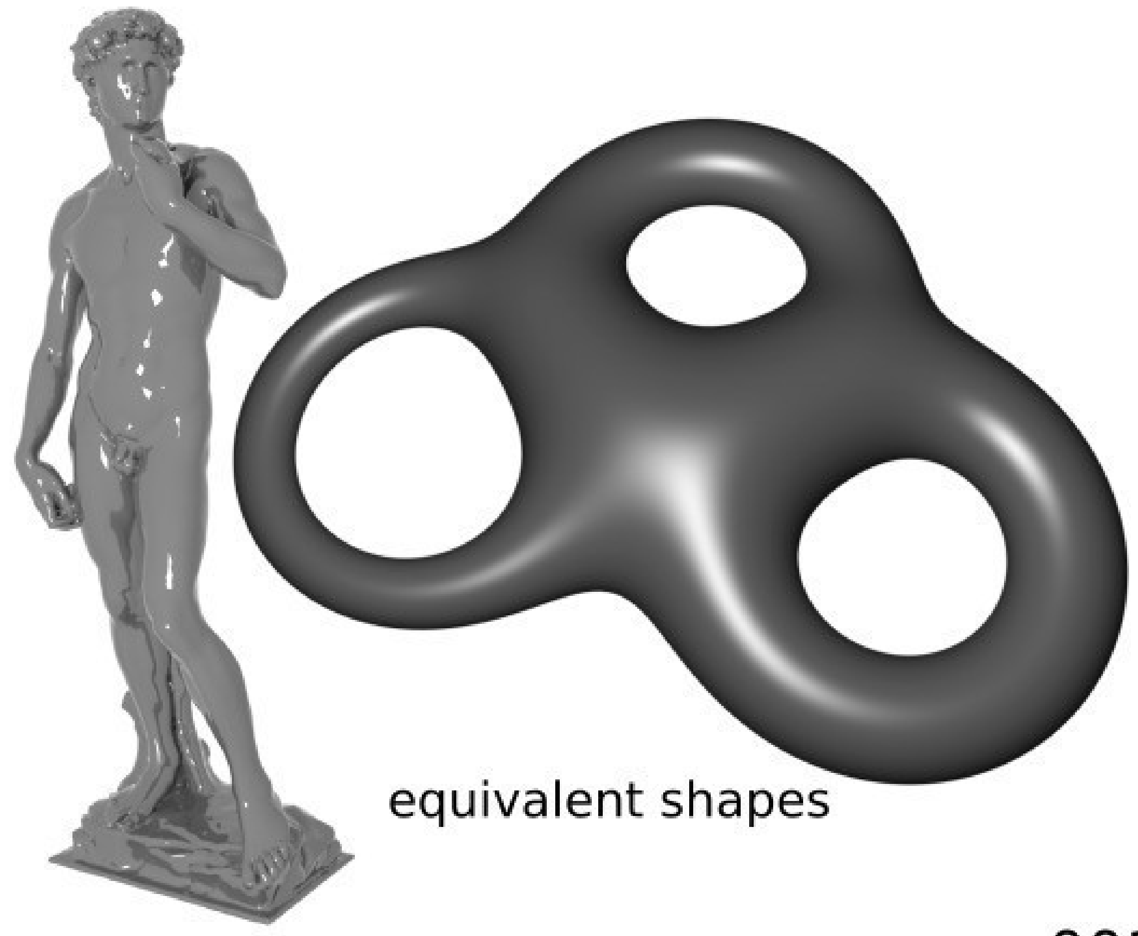
Smooth Surfaces and Differential Geometry



Topology

Topology:
Study of an object independently from its geometry

2 shapes are topologically equivalent if you can deform one onto the other using only continuous deformation (no cutting, no merging)



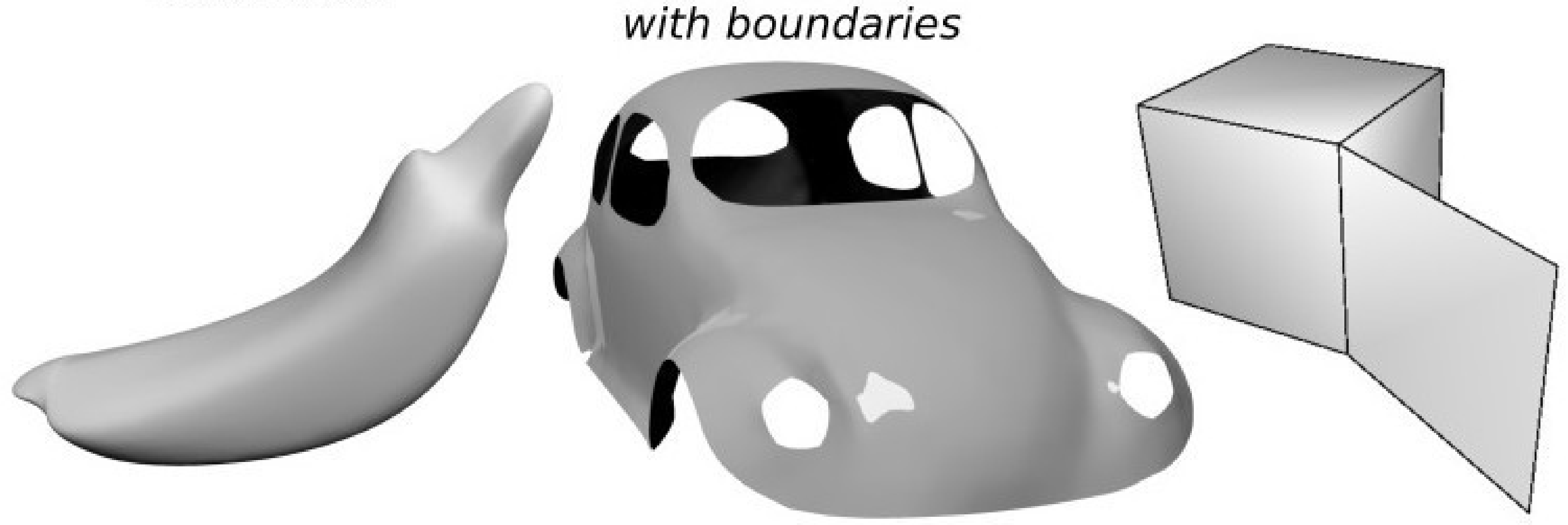
equivalent shapes

Manifold

A surface Γ is a **2-manifold** if every point has a neighbor **homeomorphic to a (half) disc**.

Homeomorphism = bijective continuous application with continuous reciprocal function.

2-manifold *2-manifold with boundaries* *not a 2-manifold*



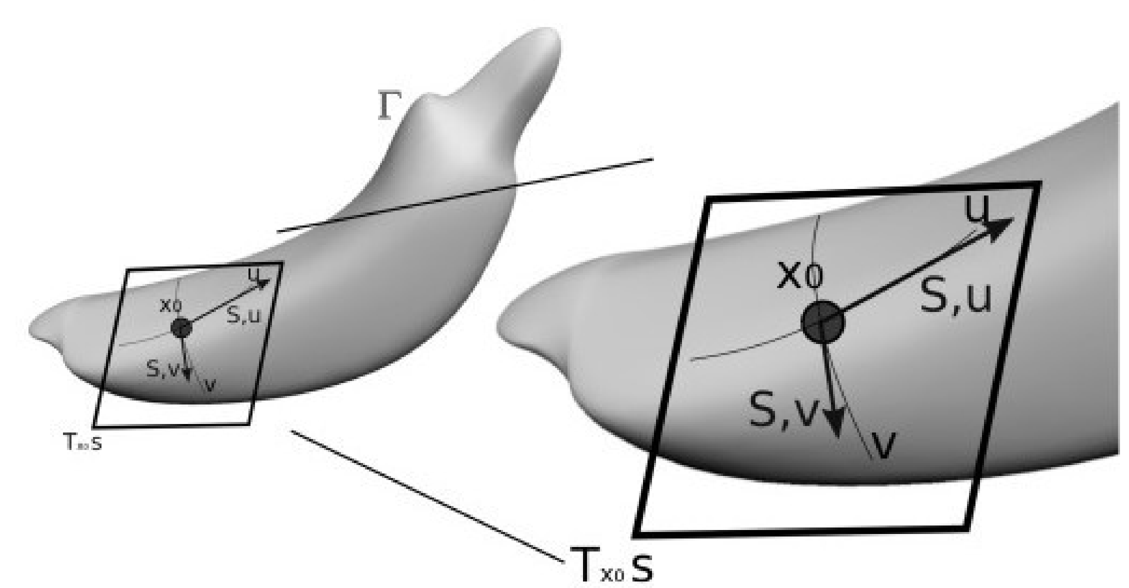
Surface continuity

Let Γ be the surface associated to the mapping S

$$S : \begin{cases} \mathcal{D} \subset \mathbb{R}^2 & \rightarrow \quad \mathbb{R}^3 \\ (u, v) & \mapsto \quad S(u, v) \end{cases}$$

$$\Gamma = tr_{\mathcal{D}}(S) = \{\mathbf{x} \in \mathbb{R}^3 | \forall (u, v) \in \mathcal{D}, S(u, v) = \mathbf{x}\}$$

S is C^1 if $S_{,u}$ and $S_{,v}$ are defined and continuous.
S is C^2 if $S_{,uu}$, $S_{,vv}$, and $S_{,uv}$ are defined and continuous.



Geometrical continuity

Γ is G^1 if there is a tangent plane in every position.
Γ is G^2 if the curvature of the surface is continuous everywhere.

Note: $S \subset \mathcal{C}^k \neq \Gamma \subset G^k$.

G^2 is necessary for the smooth reflexion.

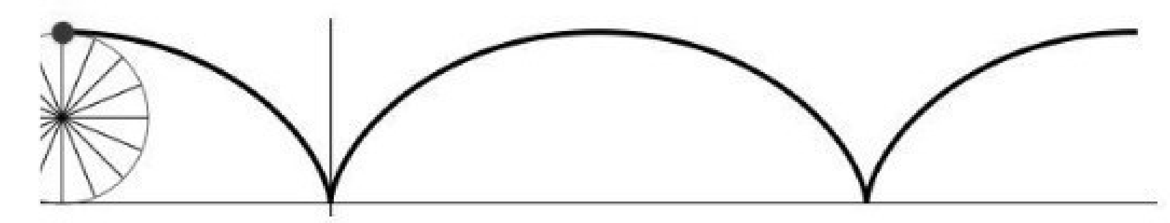
Exemple of continuity differences

$$f : \begin{cases} x(t) = t \\ y(t) = t \end{cases}, t \in [-1, 0[\qquad g : \begin{cases} x(t) = t \\ y(t) = t^2 \end{cases}, t \in [-1, 0[$$

$$\begin{cases} x(t) = t/2 \\ y(t) = t/2 \end{cases}, t \in [0, 2] \qquad \begin{cases} x(t) = t \\ y(t) = -t^2 \end{cases}, t \in [0, 1]$$

Are theses functions C^1, C^2, G^1, G^2 ?

h: $\left| \begin{array}{l} x(t) = R(t - \sin(t)) \\ y(t) = R(t - \cos(t)) \end{array} \right.$



Reminder about curves

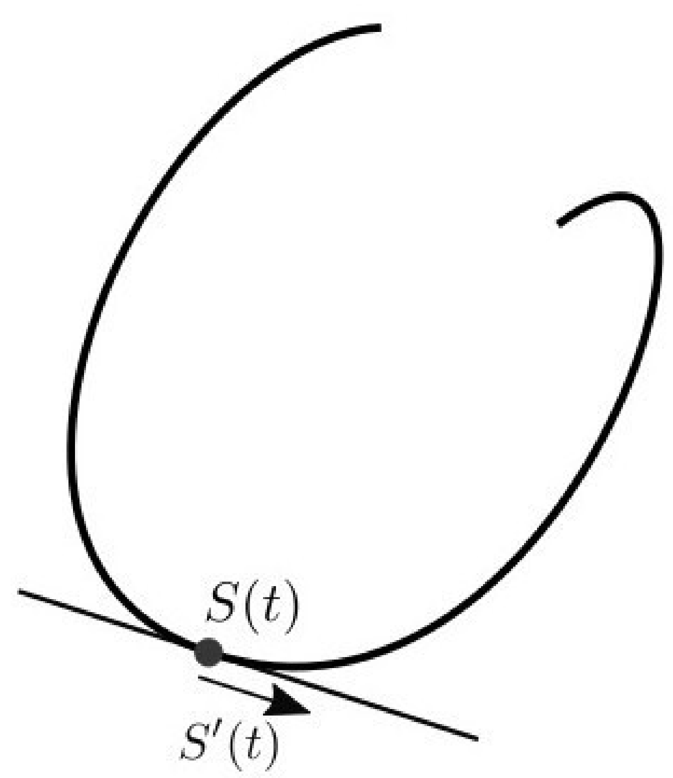
Curve $S(t)$

$\mathbb{R} \mapsto \mathbb{R}^3$

$S : t \rightarrow (S_x(t), S_y(t), S_z(t))$

Length of a curve:

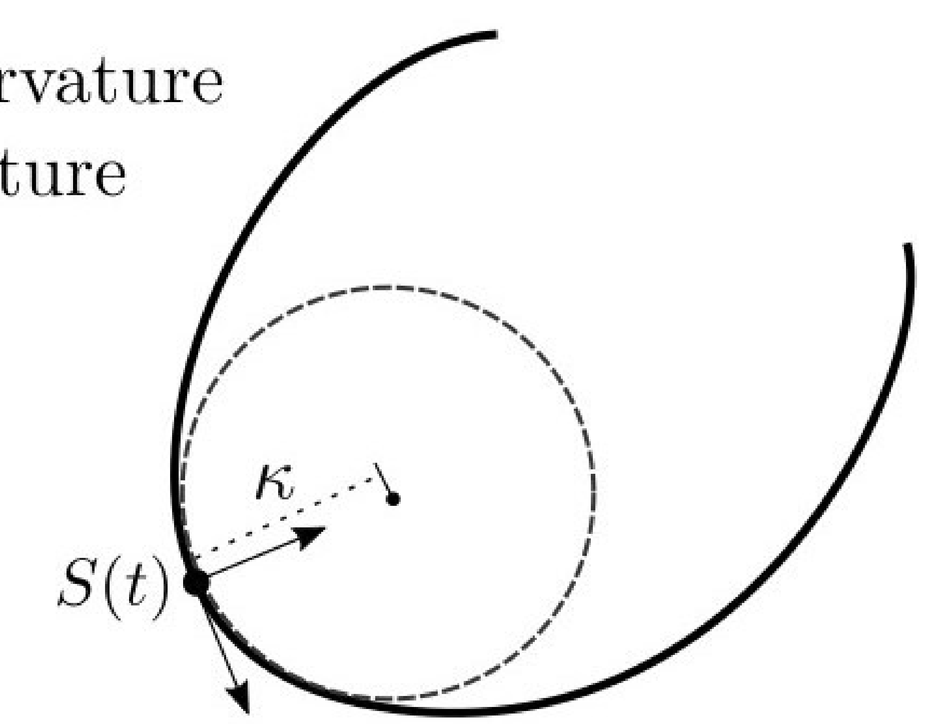
$$L = \int_t \|S'(t)\| \, \mathrm{d}t$$



Curvature of a curve

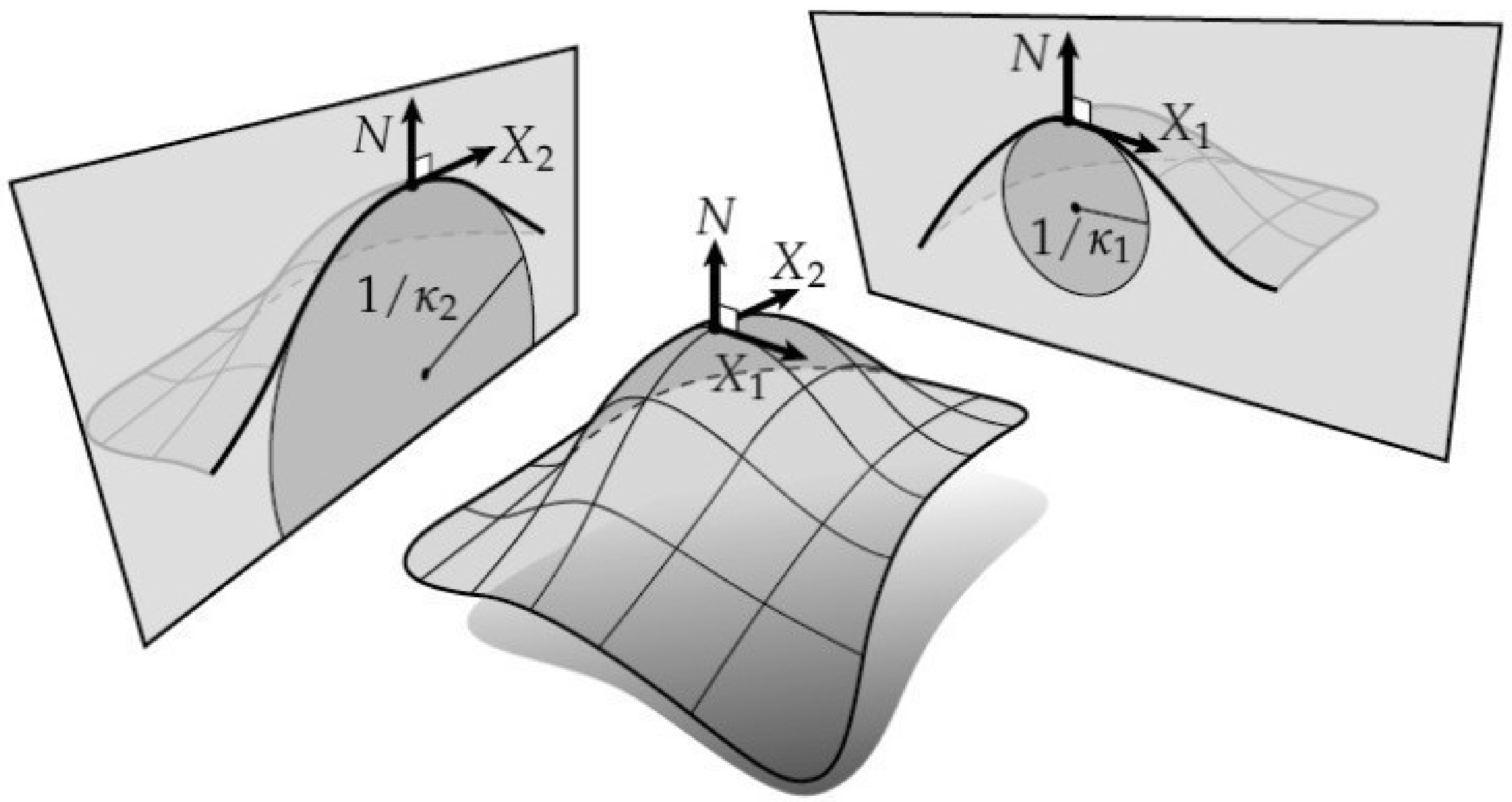
κ is the radius of curvature
$\lambda = 1/\kappa$ is the curvature

$$\left| \begin{array}{l} \kappa = S'(\rho) \\ \rho = \|S'(t)\| \end{array} \right.$$



$$\kappa = \frac{1}{\|S'(t)\|^3} \left(\|S'(t)\|^2 \, \|S''(t)\|^2 - < S'(t), S''(t) >^2 \right)^{1/2}$$

Intuition for the curvature on a Surface



[Keenan Crane, Digital Geometry Processing with Discrete Exterior Calculus, SIGGRAPH 2013]

Tangent plane

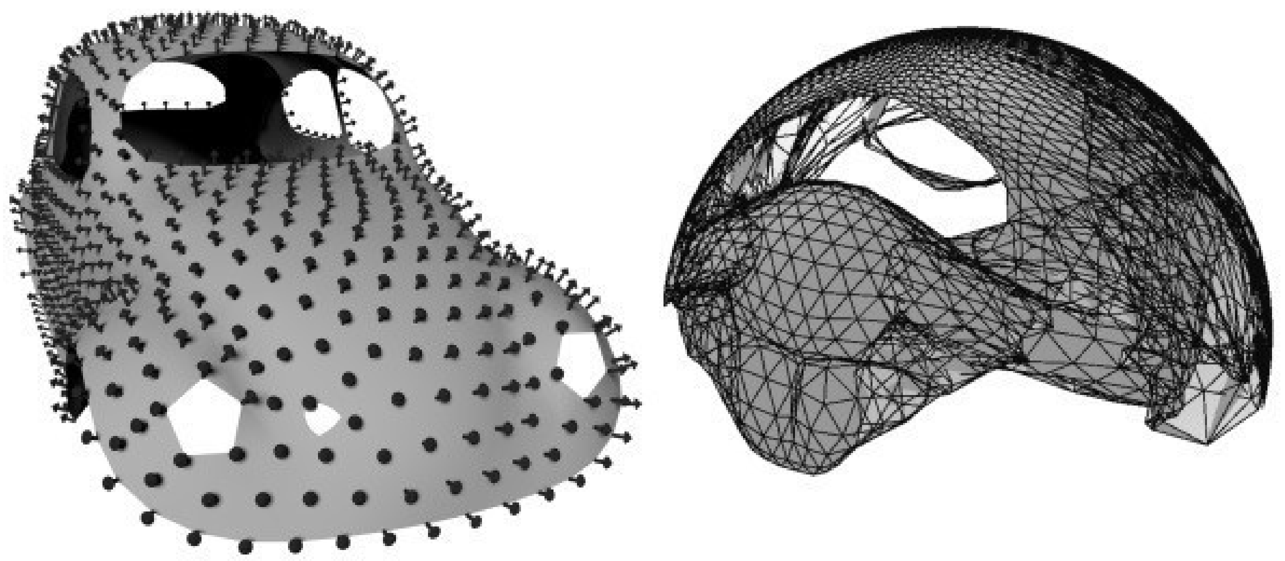
Surface in $\mathbb{R}^3$: $S(u,v) = (S_x(u,v), S_y(u,v), S_z(u,v))$

Normal: $n(u,v) = (S_{,u} \times S_{,v}) / \|S_{,u} \times S_{,v}\| \ \in \mathbb{S}^2$

Tangent space of S in $x_0 = S(u,v)$

$$T_{x_0}S = \mathrm{Im}(\mathrm{D}\,S(u,v)) = \{S_{,u}(u,v)\,h_u + S_{,v}(u,v)\,h_v | (h_u,h_v) \in \mathbb{R}^2\}$$

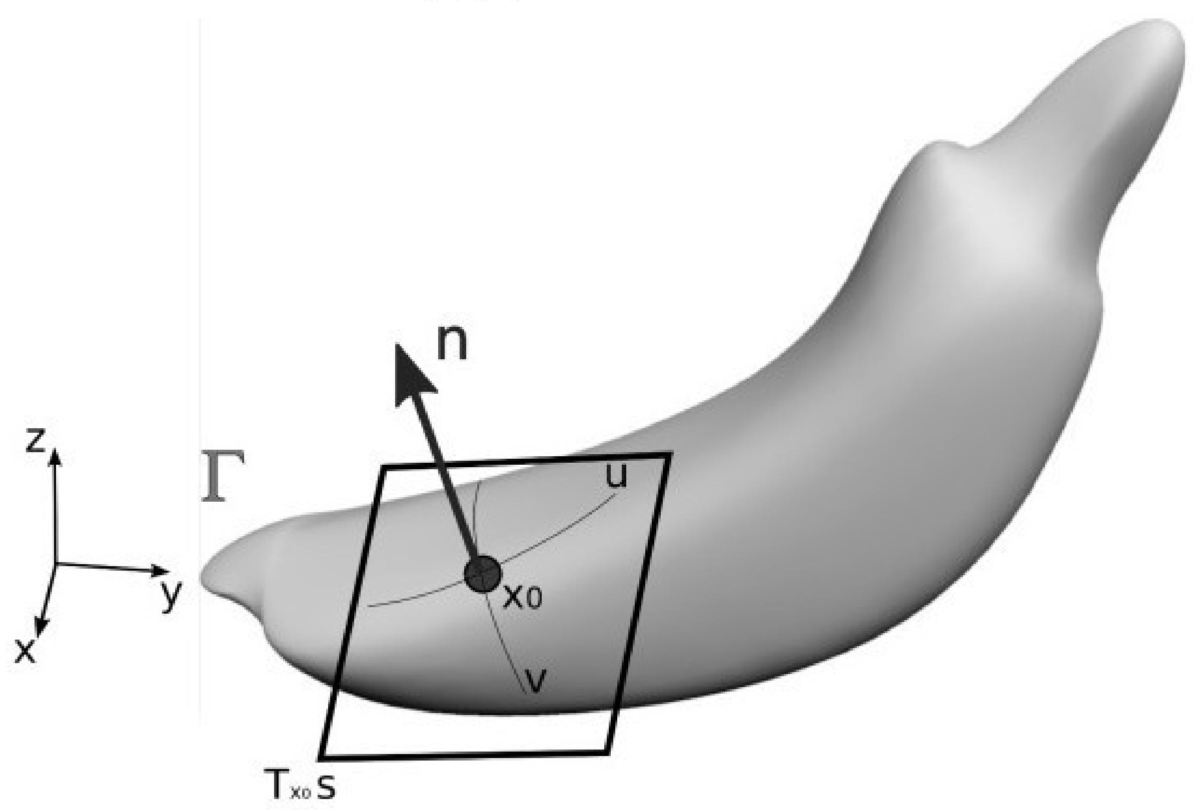
$$N : \begin{cases} \Gamma & \rightarrow \quad \mathbb{S}^2 \\ x_0 = S(u,v) & \mapsto \quad N(x_0) = n(u,v) \end{cases}$$



Integral properties

Area of Γ : $\displaystyle\iint_{(u,v)\in\mathcal{D}} \|S_{,u} \times S_{,v}\| \,\mathrm{d}u\,\mathrm{d}v$

Volume defined by Γ $\displaystyle\iint_{(u,v)\in\mathcal{D}} S_z(u,v)\, n_z^S(u,v)\,\mathrm{d}u\,\mathrm{d}v$



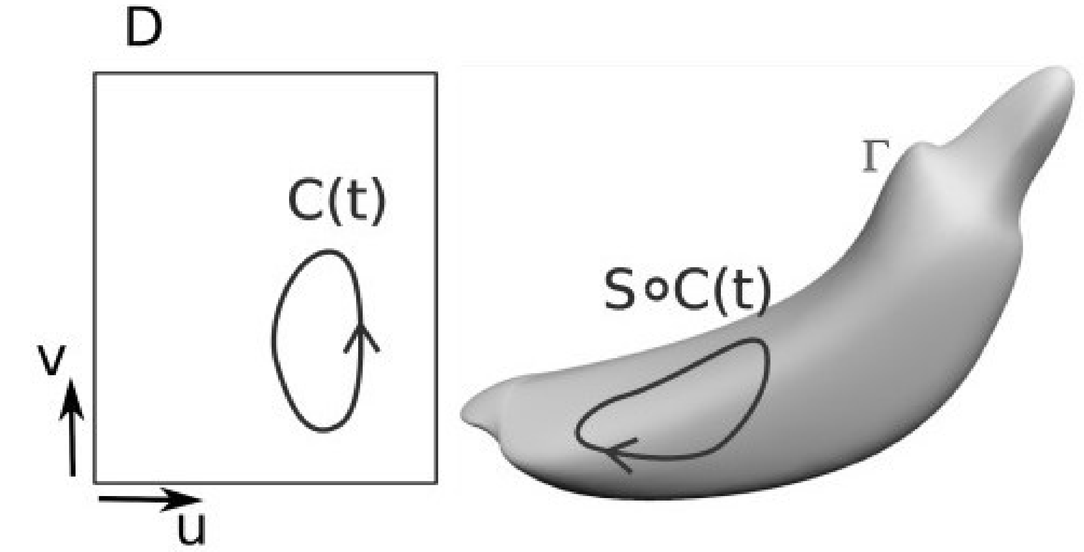
First fundamental form

Curve $C \subset \mathcal{D}$ of length $L = \displaystyle\int_t < C'(t), C'(t) >^{1/2} \,\mathrm{d}t$

Length of curve $C_s = S(C_x, C_y)$

$$L_s = \int_t < (S \circ C)'(t), (S \circ C)'(t) >^{1/2} \,\mathrm{d}t$$

$$= \int_t (C'^T(t)\,\mathrm{I}_S(t)\,C'(t))^{1/2} \,\mathrm{d}t$$



First fundamental form

Derivation $L_s = \int_t \left((S \circ C)'^T(t)\ (S \circ C)'(t) \right)^{1/2}\ \mathrm{d}t$

$$(S \circ C)' = C'_x\,(S_{,u} \circ C) + C'_y\,(S_{,v} \circ C)$$

$$(S \circ C)' = (S_{,u}\ \ S_{,v}) \begin{pmatrix} C'_x \\ C'_y \end{pmatrix} = \partial S^T\, C'$$

$$\begin{aligned}(S \circ C)'^T\ (S \circ C)' &= (C'^T \partial S)\,(\partial S^T\, C') \\ &= C'^T (\partial S\, \partial S^T)\, C' \\ &= \boxed{C'^T \mathrm{I}_s\, C'}\end{aligned}$$

First fundamental form

I_S First fundamental form / metric tensor

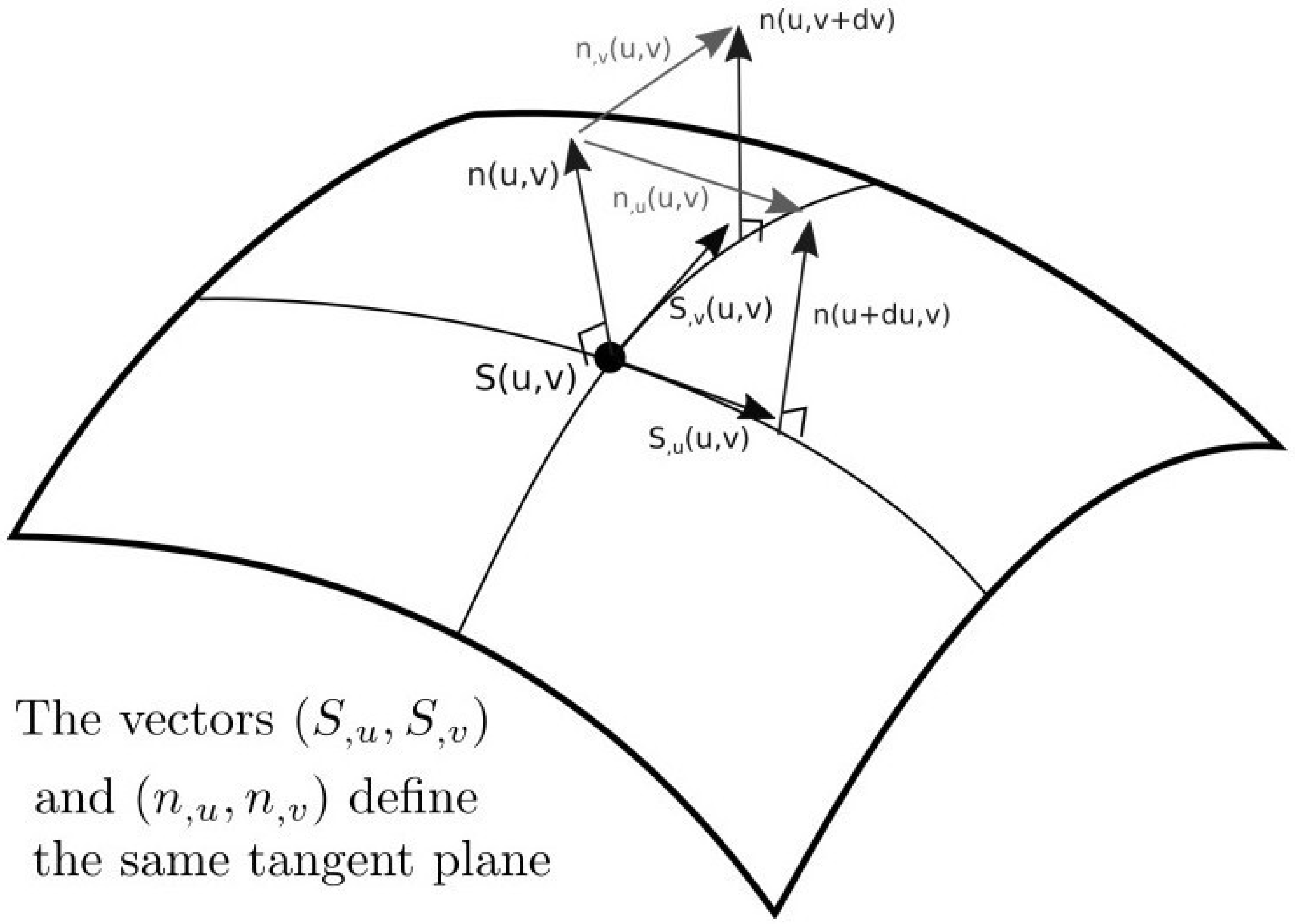
$$\mathrm{I}_S = \begin{pmatrix} S_{,u}^2 & < S_{,u}, S_{,v} > \\ < S_{,u}, S_{,v} > & S_{,v}^2 \end{pmatrix}$$

I_S quadratic form associated to $< \mathrm{d}S, \mathrm{d}S >$

$\sqrt{\det(\mathrm{I}_S)}$ = local area change

Area of $\Gamma = \iint_{(u,v) \in \mathcal{D}} \sqrt{\det(\mathrm{I}_S)}\, \mathrm{d}u\, \mathrm{d}v$

Derivative of the normals



The vectors $(S_{,u}, S_{,v})$ and $(n_{,u}, n_{,v})$ define the same tangent plane

Second fundamental form

The vectors $(n_{,u}, n_{,v})$ can be expressed in the $(S_{,u}, S_{,v})$ plane

$$\begin{cases} n_{,u} = w_{00}\, S_{,u} + w_{01}\, S_{,v} \\ n_{,v} = w_{10}\, S_{,u} + w_{11}\, S_{,v} \end{cases} \qquad \begin{pmatrix} n_{,u}^T \\ n_{,v}^T \end{pmatrix} = \mathrm{W}_S \begin{pmatrix} S_{,u}^T \\ S_{,v}^T \end{pmatrix}$$

In multiplying both sides by $(S_{,u}, S_{,v})$

$$\mathrm{II}_S = \mathrm{W}_S\, \mathrm{I}_S$$

Where II_S is the second fundamental form associated to S

$$\mathrm{II}_S = \begin{pmatrix} < n_{,u}, S_{,u} > & < n_{,u}, S_{,v} > \\ < n_{,v}, S_{,u} > & < n_{,v}, S_{,v} > \end{pmatrix}$$

Relation with second derivatives

Using orthogonality

$$\begin{cases} <n, S_{,u}> = 0 \\ <n, S_{,v}> = 0 \end{cases}$$

Differentiating

$$\begin{cases} <n_{,u}, S_{,u}> = -<n, S_{,uu}> \\ <n_{,v}, S_{,u}> = -<n, S_{,uv}> \\ <n_{,v}, S_{,v}> = -<n, S_{,vv}> \end{cases}$$

II can be expressed from n and the second derivatives of S

$$\mathrm{II}_S = -\begin{pmatrix} <n, S_{,uu}> & <n, S_{,uv}> \\ <n, S_{,uv}> & <n, S_{,vv}> \end{pmatrix}$$

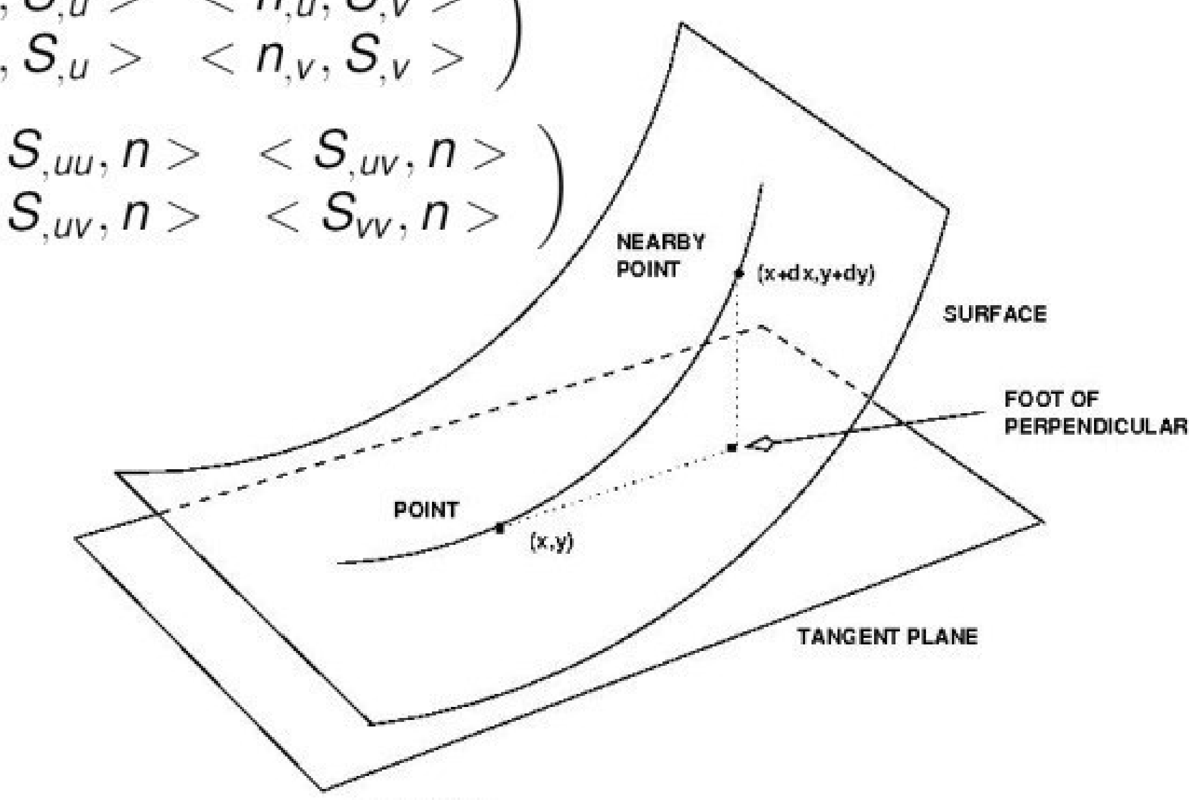
Second fundamental form

II is the quadratic form associated to $<\mathrm{s}S, \mathrm{d}n>$

=Taylor expension of S in its tangent plane

$$\mathrm{II}_S = \begin{pmatrix} <n_{,u}, S_{,u}> & <n_{,u}, S_{,v}> \\ <n_{,v}, S_{,u}> & <n_{,v}, S_{,v}> \end{pmatrix}$$

$$\Leftrightarrow \mathrm{II}_S = -\begin{pmatrix} <S_{,uu}, n> & <S_{,uv}, n> \\ <S_{,uv}, n> & <S_{,vv}, n> \end{pmatrix}$$



[Wikipedia]

Weingarten application

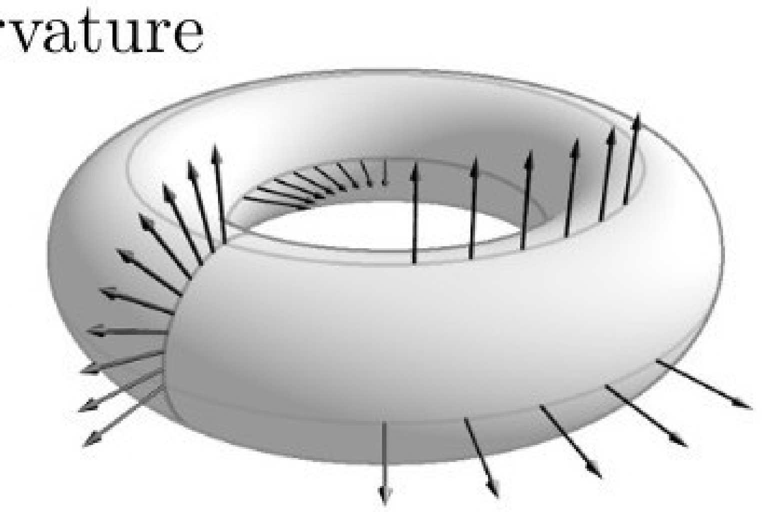
The matrix W_S such that $\mathrm{W}_S = \mathrm{II}_S\ \mathrm{I}_S^{-1}$
is the Weingarten matrix (or Shape operator)

W_S is diagonalizable and has real eigenvalues

$$\mathrm{W}_S = \mathrm{V}^T\ \Lambda\ \mathrm{V} \quad \text{with} \quad \Lambda = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$$

λ_1, λ_2 are the principal curvature

Weingarten map =
differential of the Gauss map

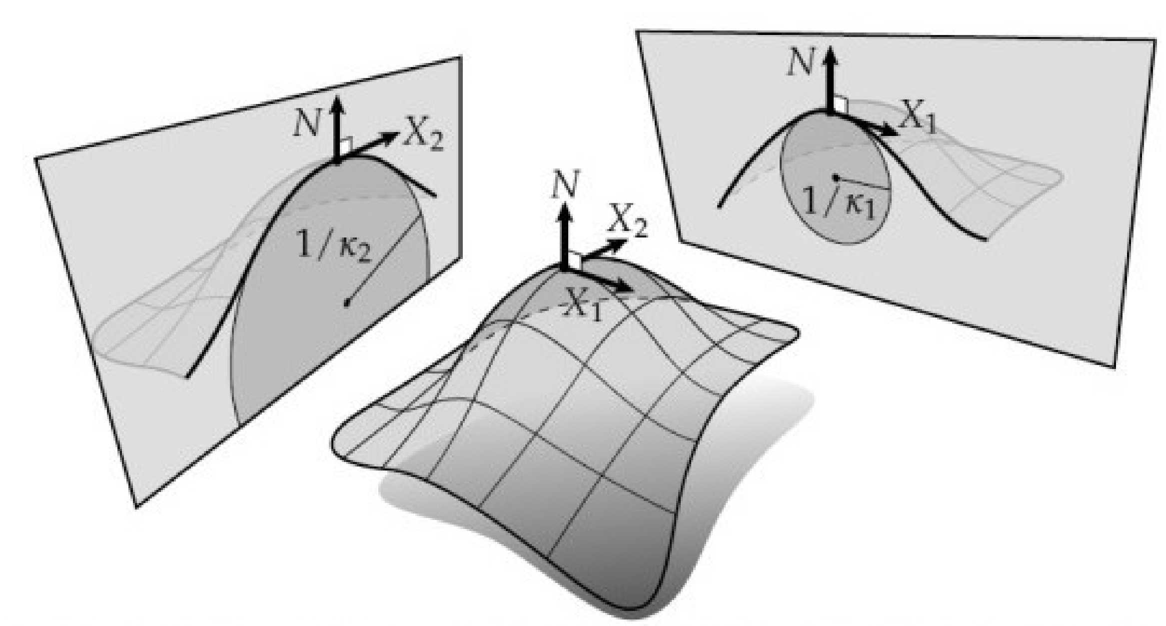


Principal curvatures

Eigenvalues of W_S : principal curvatures (λ_1, λ_2).

Principal radius of curvatures $(\kappa_1 = \lambda_1^{-1}, \kappa_2 = \lambda_2^{-1})$.

Eigenvectors of W_S :
direction $(\mathbf{v}_1, \mathbf{v}_2)$ of the principal curvatures.



[Keenan Crane, Digital Geometry Processing with Discrete Exterior Calculus, SIGGRAPH 2013]

Curvature types

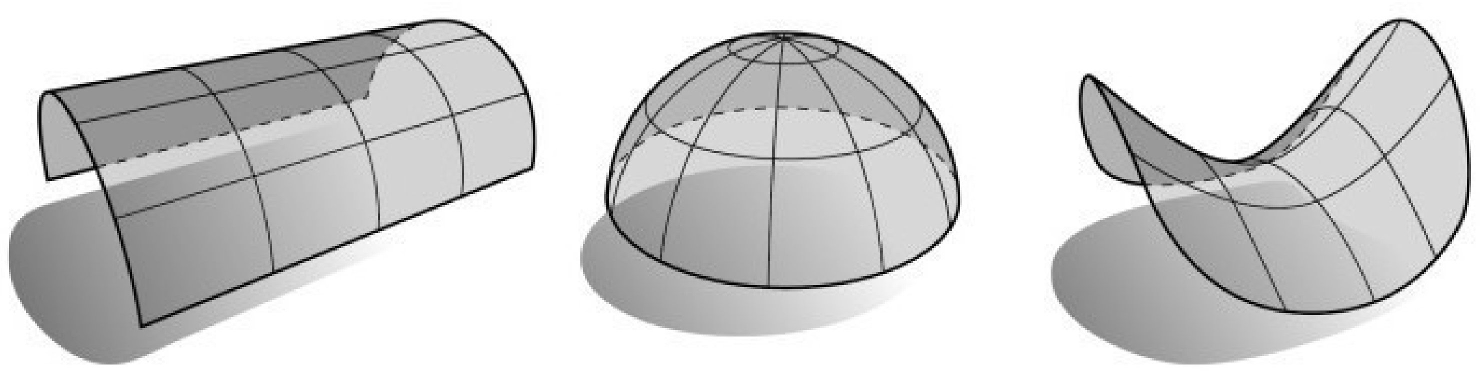
A surface can be locally

Planar: $\lambda_1 = \lambda_2 = 0$

Cylindrical: $\lambda_i \neq 0, \lambda_j = 0$

Elliptic: $\lambda_i \, \lambda_j > 0$

Hyperbolic: $\lambda_i \, \lambda_j < 0$



[Keenan Crane, Digital Geometry Processing with Discrete Exterior Calculus, SIGGRAPH 2013]

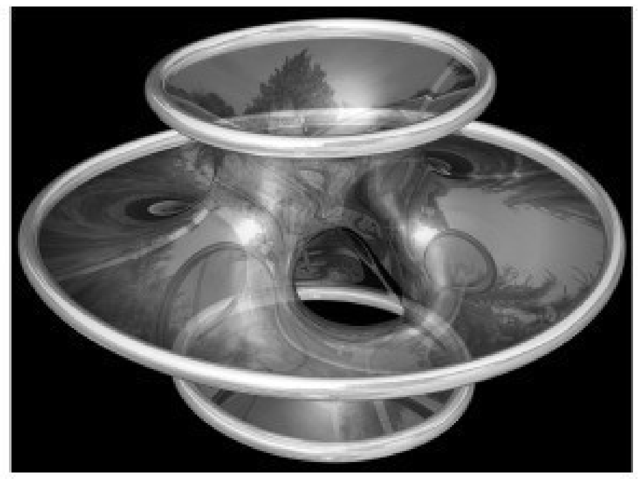
Gauss and mean curvatures

Gauss curvature: $K_s = \lambda_1 \, \lambda_2 = \det(\mathrm{W}_S) = \dfrac{\det(\mathrm{II}_S)}{\det(\mathrm{I}_S)}$

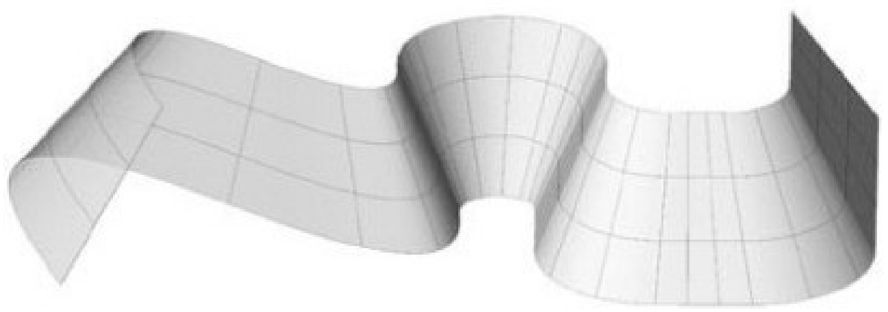
Mean curvature: $H_S = \dfrac{1}{2}(\lambda_1 + \lambda_2) = \dfrac{1}{2}\mathrm{tr}\,(\mathrm{W}_S)$

$H_S = 0 \Leftrightarrow$ S is a minimal surface

$K_S = 0 \Leftrightarrow$ S is a developable surface



[Paul Nylander]



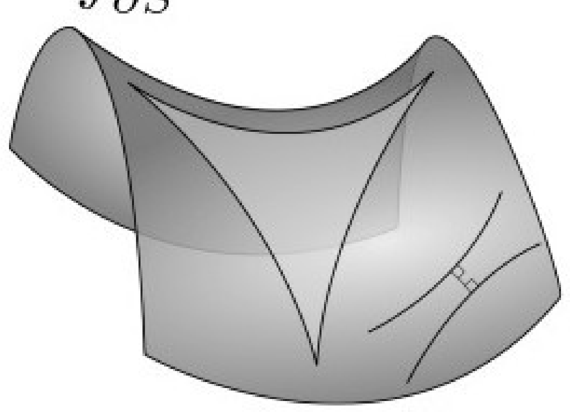
[M. Nettelbladt]

Integral relation

Gauss-Bonnet theorem: $\int_S K_s \, \mathrm{d}A + \int_{\partial S} k_g \, \mathrm{d}s = 2\pi\chi(S)$

k_g: geodesic curvature

χ: Euler characteristic (topological invariant)



[Wikipedia]

Application to a mesh around a vertex

$$K \simeq \frac{1}{A}\left(\sum_i \theta_i - 2\pi\right)$$