Numerical solution of ODE
(Ordinary Differential Equation)

What is an ODE ?

Example:

f is an unknown **function**

$$f'(x) = af(x) + b$$

dérivative of f
depends of f

f is 1D function

(otherwise PDE)

Can also be written as: $y' = ay + b$

What is an ODE ?

More examples:

Linear, constant coefficients

$$f'(x) = 4f(x) + 2$$

Linear, variable coefficients

$$f'(x) = 4(x-5)f(x) + 2x^2 - 7$$

Non linear

$$f'(x) = 4x\sin(xf(x)) + 2/f^2(x)$$

General expression

$$\boxed{f'(x) = \mathcal{F}(x, f)}$$

Even more general: Implicit formulation

$$\mathcal{R}(x, f, f') = 0$$

What is an ODE?

Order of ODE:

Second order, linear
$$f''(x) = 2f'(x) + 3f(x) - 4$$

Third order, non linear
$$f^{(3)}(x) = 2x^2 \sin\left(f''(x) + f'(x)\right) - f^2(x)$$

General definition:
$$\boxed{f^{(n)}(x) = \mathcal{F}(x, f, f', \ldots, f^{(n-1)})}$$

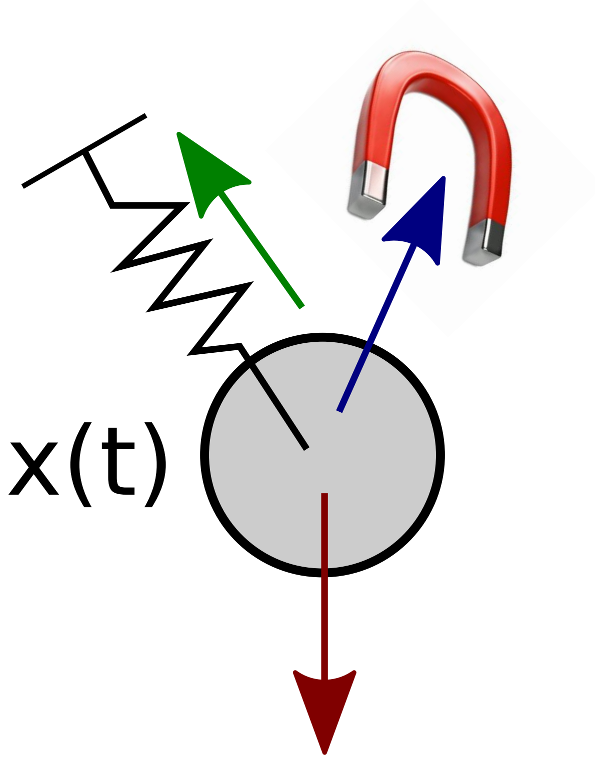
Even more general:
$$\mathcal{R}(x, f, f', \ldots, f^{n-1}, f^{(n)}) = 0$$

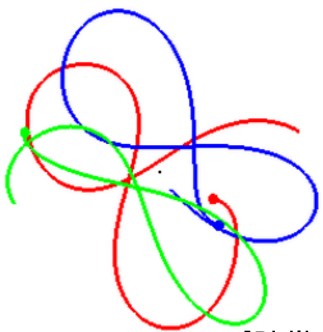
Why do we need ODE?

Physics:

$$m\,a(t) = \sum F(x(t), t)$$

$$x''(t) = \frac{1}{m} F(x(t))$$





[Philippe Roux]

Why do we need ODE?

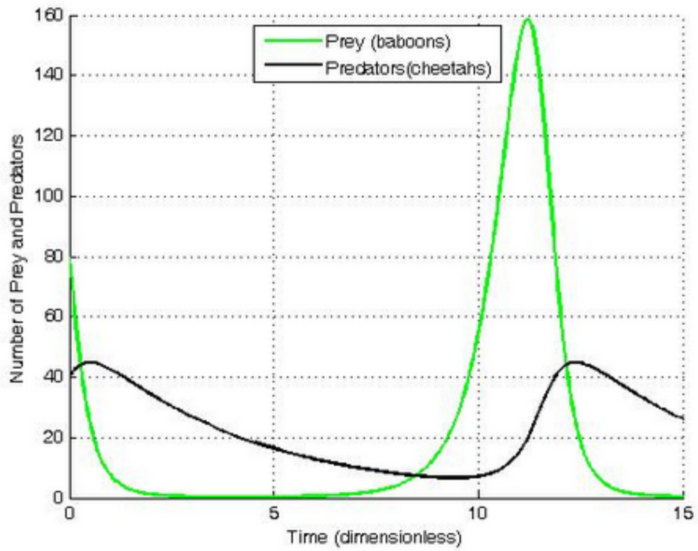
Biology:

Population grows

f1 : prey
f2 : predator

$$\begin{cases} f_1'(t) = f_1(t)(\alpha - \beta f_2(t)) \\ f_2(t) = -f_2(t)(\gamma - \delta f_1(t)) \end{cases}$$

Lokta-Volterra equation

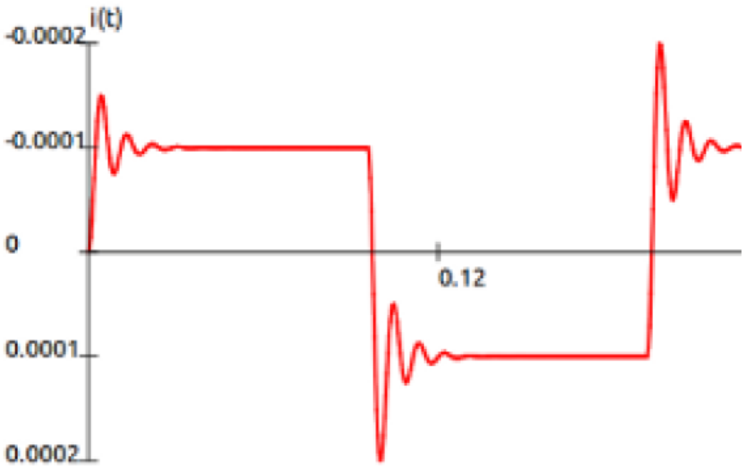


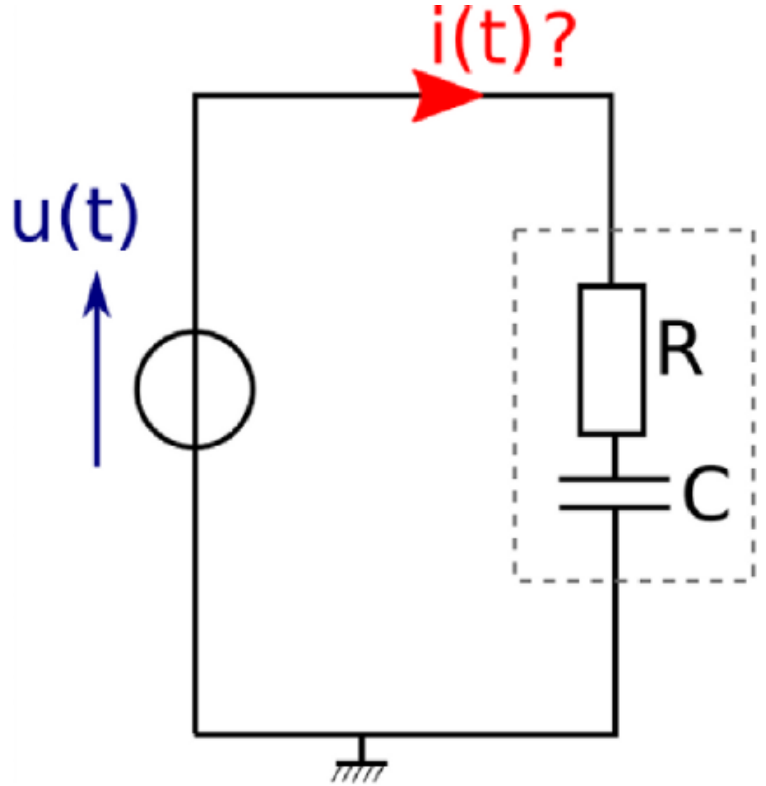
[Wikipedia]

Why do we need ODE?

Physics:

$$RCi'(t) + i(t) = Cu'(t)$$





What can we solve analytically?

Linear + constant coefficient

$$a_0 f(x) + a_1 f'(x) + \cdots + a_n f^{(n)}(x) = r(x)$$

Linear + variable coefficient + low order

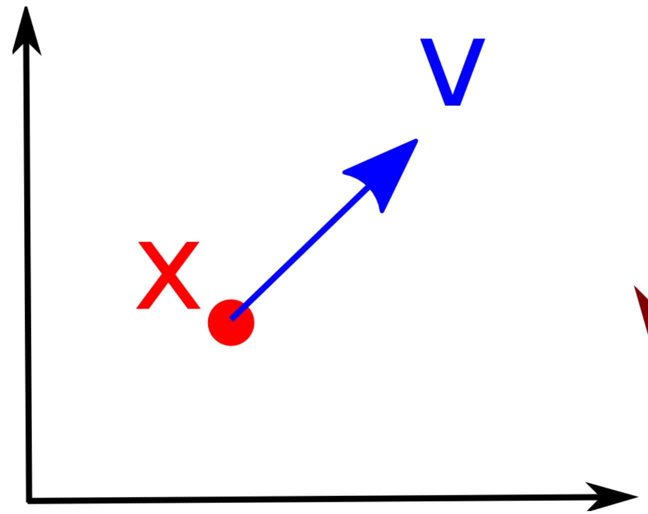
$$a_0(x) f(x) + a_1(x) f'(x) + a_2(x) f''(x) = r(x)$$

Non linear: Almost never

Existence and uniqueness ?

Case study: Free fall under gravity

Motion equations:



Initial conditions:

$$x(t=0)=x_0$$
$$v(t=0)=v_0$$

Numerical approach

Motion equations:

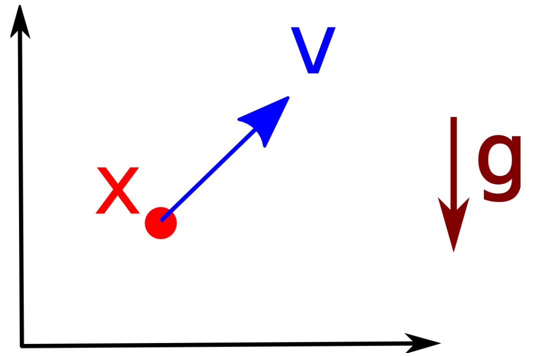
x'(t)=v(t)

v'(t)=g

Initial conditions:

$x(t=0)=x_0$

$v(t=0)=v_0$



Approximation:

$$f'(t) \simeq \frac{f(t+\mathrm{d}t)-f(t)}{\mathrm{d}t}$$

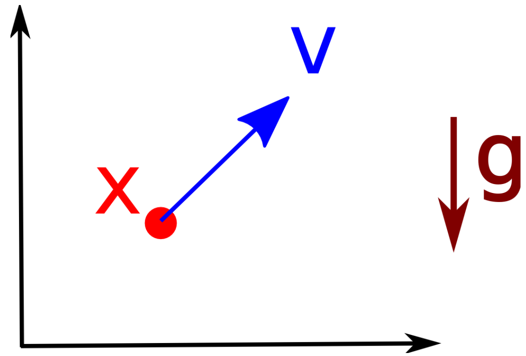
Numerical approach

Motion equations:

$x'(t)=v(t)$
$v'(t)=g$

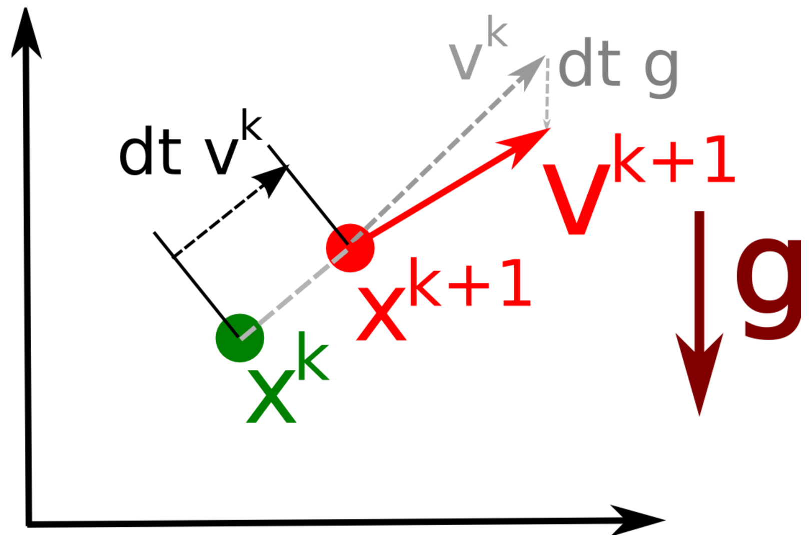
Initial conditions:

$x(t=0)=x_0$
$v(t=0)=v_0$



Solution

$$\begin{cases} v^{k+1} = v^k + (\Delta t)g \\ x^{k+1} = x^k + (\Delta t)v^k \end{cases}$$



Code;

```
x=x0;
v=v0
for(k=0;k<N;++k)
{
  x=x+dt*v;
  v=v+dt*g;
}
```

Numerical approach

Motion equations:

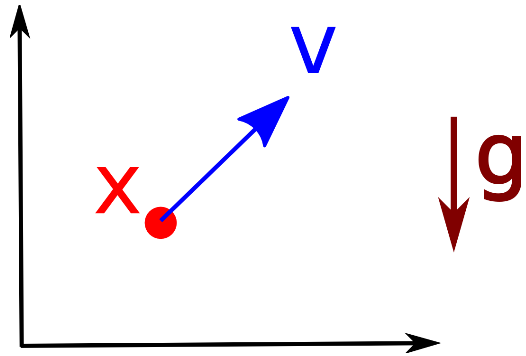
x'(t)=v(t)

v'(t)=g

Initial conditions:

$x(t=0)=x_0$

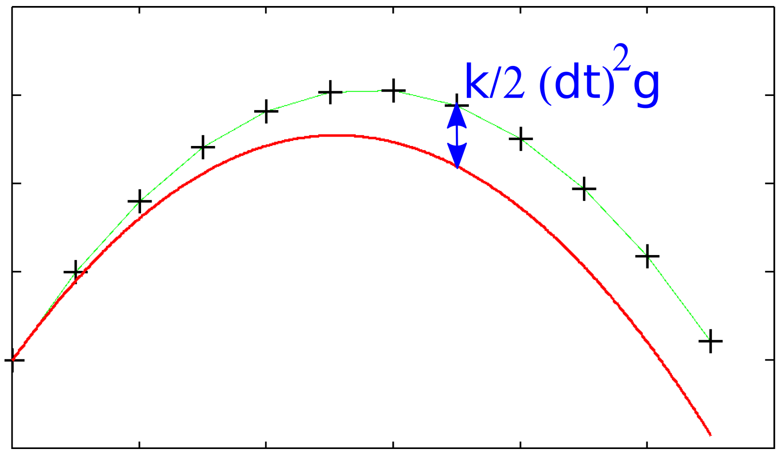
$v(t=0)=v_0$



Numerical solution:

$$\begin{cases} v^{k+1} = v^k + (\Delta t)g \\ x^{k+1} = x^k + (\Delta t)v^k \end{cases}$$

$$\Rightarrow \begin{cases} x^{k+2} = 2x^{k+1} - x^k + (\Delta t)^2 g \\ x^0 = x_0 \\ x^1 = x_0 + \Delta t v_0 \end{cases}$$



$$\Rightarrow \quad x(t = k\Delta t) = x_0 + (k\Delta t)\, v_0 + \frac{k(k-1)}{2}(\Delta t)^2 g$$

Real solution:

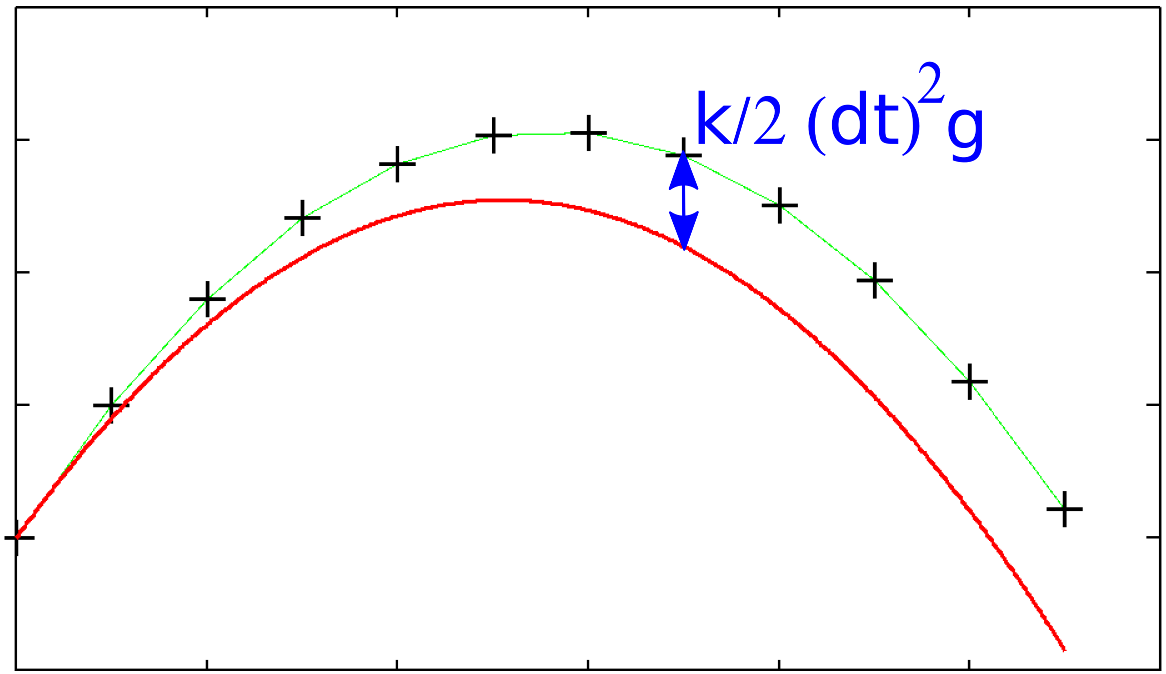
$$\tilde{x}(t = k\Delta t) = x_0 + (k\Delta t) v_0 + \frac{1}{2}(k\Delta t)^2 g$$

Accuracy

Numerical error: $\|x(k\Delta t) - \tilde{x}(k\Delta t)\| = \frac{k}{2}(\Delta t)^2 g$

Definition of accuracy of order h:

$$\|x(k\Delta t) - \tilde{x}(k\Delta t)\| = \mathcal{O}((\Delta t)^{h+1})$$



Matrix formulation

Linear ODE of order n
= system of ODE of order 1

$$u(t) = \begin{pmatrix} x(t) \\ x'(t) \end{pmatrix}$$

$$u'(t) = \mathrm{A}u(t) + \mathrm{b}(t)$$

with $\mathrm{A} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ $\mathrm{b}(t) = \begin{pmatrix} 0 \\ g \end{pmatrix}$ $u(t) = \begin{pmatrix} x(t) \\ v(t) \end{pmatrix}$

Matrix formulation

$$u'(t) = \mathrm{A}u(t) + \mathrm{b}(t)$$

$$u(t) = \begin{pmatrix} x(t) \\ v(t) \end{pmatrix} \qquad \mathrm{A} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

$$\mathrm{b}(t) = \begin{pmatrix} 0 \\ g \end{pmatrix}$$

Same approach:

$$u^{k+1} = (\mathrm{I} + \Delta t\ \mathrm{A})\, u^k + \Delta t\ \mathrm{b}$$

Same solution ...

Matrix formulation

$$u'(t) = \mathrm{A}u(t) + \mathrm{b}(t) \qquad u(t) = \begin{pmatrix} x(t) \\ v(t) \end{pmatrix} \qquad \mathrm{A} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

$$\mathrm{b}(t) = \begin{pmatrix} 0 \\ g \end{pmatrix}$$

?

Explicit Euler: $$\frac{u^{k+1} - u^k}{\Delta t} = \mathrm{A}\, u^k + \mathrm{b}$$

Matrix formulation

$$u'(t) = \mathrm{A}u(t) + \mathrm{b}(t) \qquad u(t) = \begin{pmatrix} x(t) \\ v(t) \end{pmatrix} \qquad \mathrm{A} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

$$\mathrm{b}(t) = \begin{pmatrix} 0 \\ g \end{pmatrix}$$

?

Explicit Euler: $$\frac{u^{k+1} - u^k}{\Delta t} = \mathrm{A}\, u^k + \mathrm{b}$$

Implicit Euler: $$\frac{u^{k+1} - u^k}{\Delta t} = \mathrm{A}\, u^{k+1} + \mathrm{b}$$

now
unknown

Implicit Euler

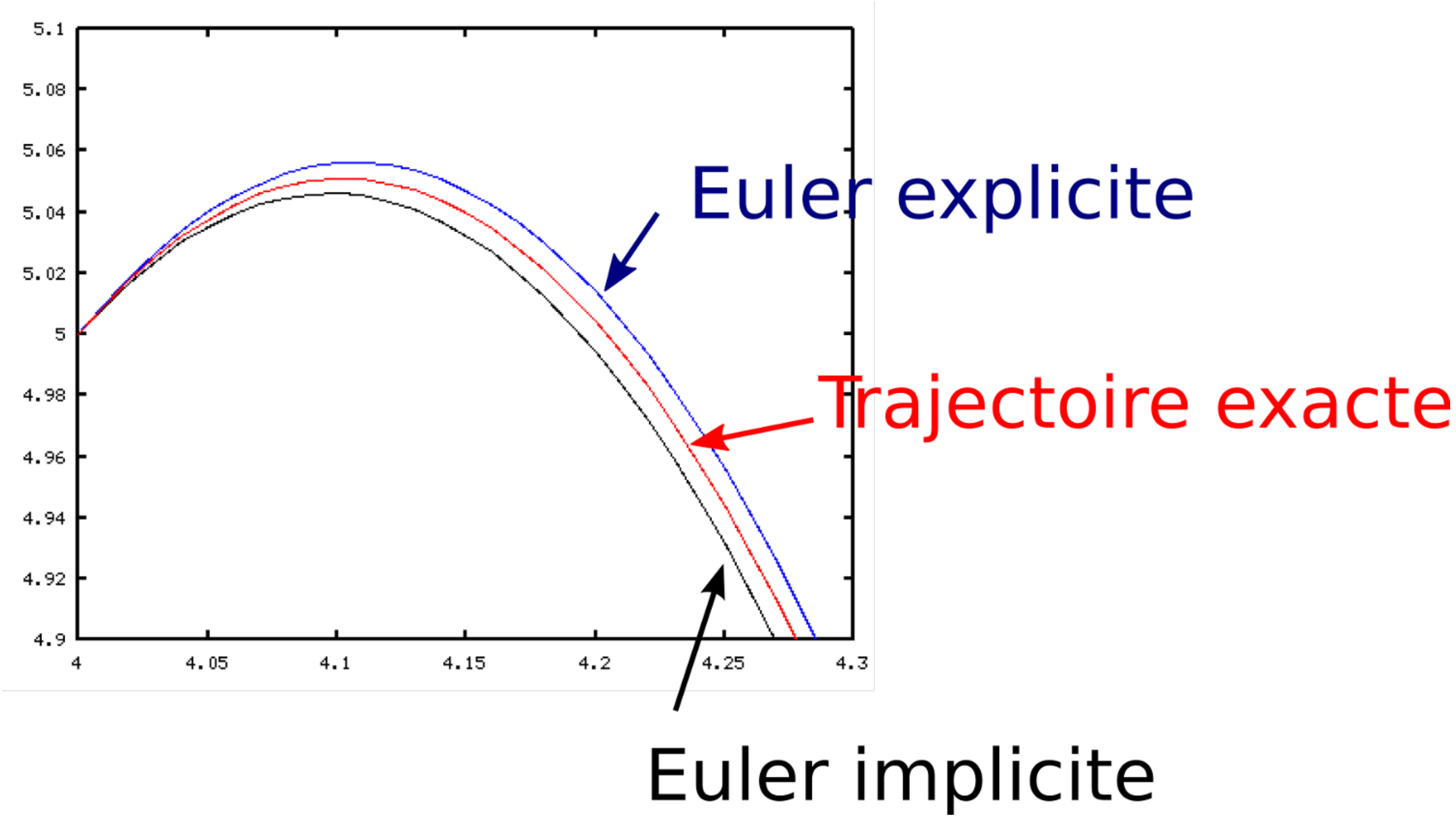
$$u^{k+1} = (\mathrm{I} - \Delta t \mathrm{A})^{-1} \left(u^k + \Delta t \, \mathrm{b}\right)$$

$$\begin{cases} x^{k+2} = 2x^{k+1} - x^k + (\Delta t)^2 g \\ x^0 = x_0 \\ x^1 = x_0 + \Delta t \, v_0 + (\Delta t)^2 g \end{cases}$$

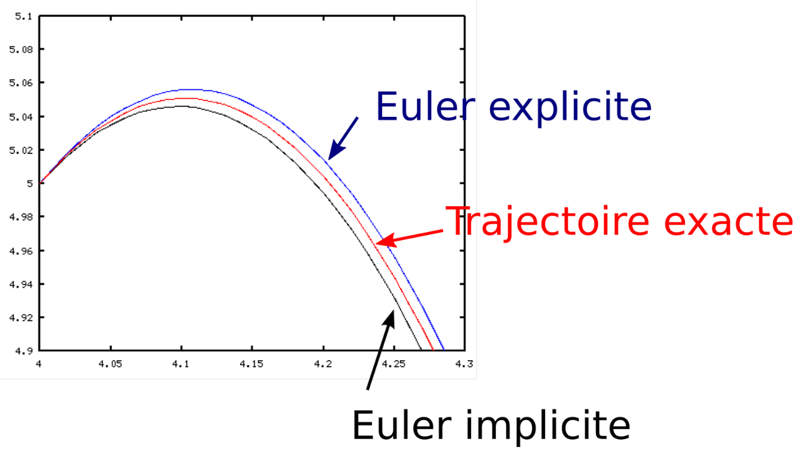
$$x(k\Delta t) = x_0 + (k\Delta t)\, v_0 + \frac{k(k+1)}{2}(\Delta t)^2 g$$

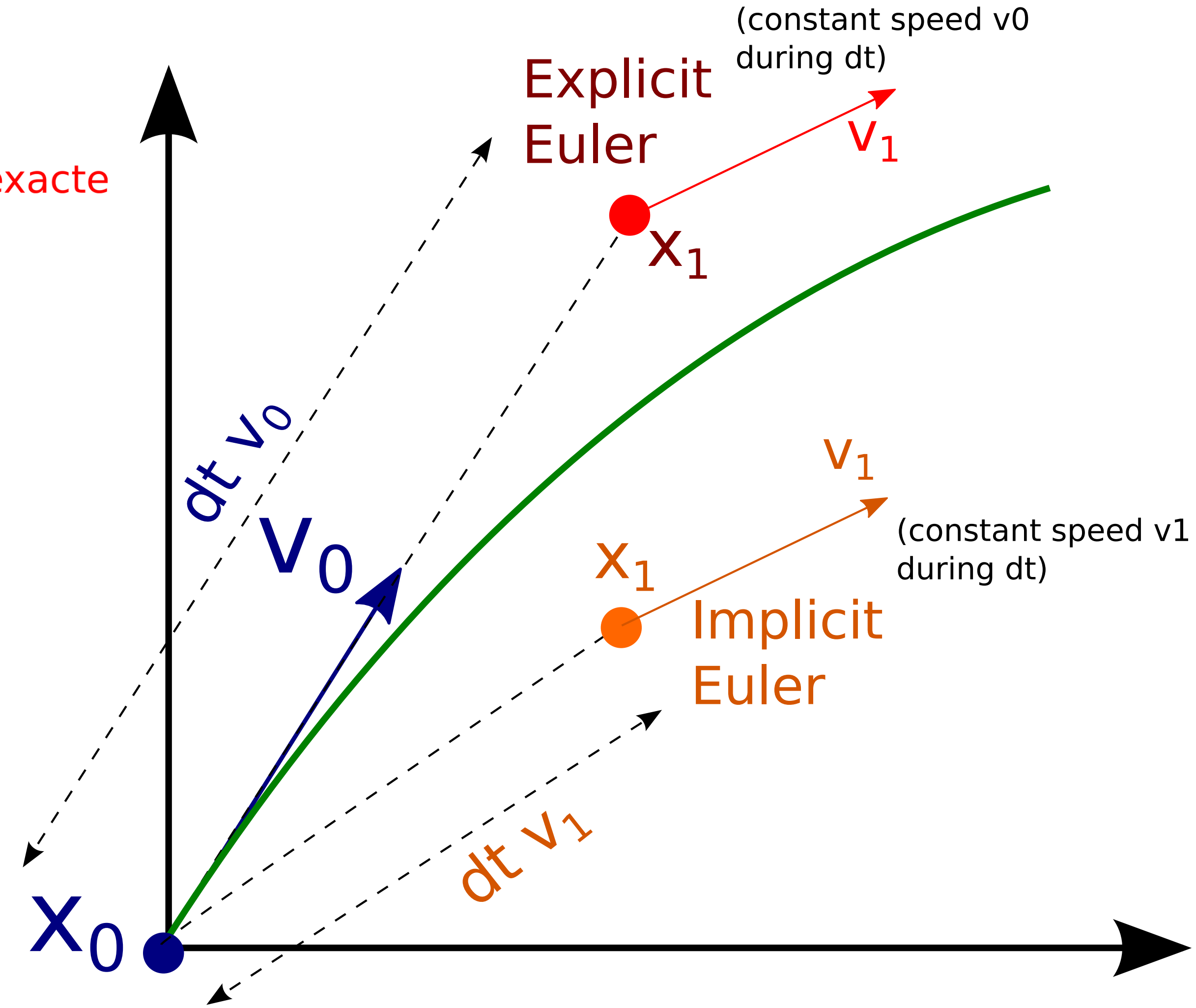
Still not the true solution

Implicit Euler



Implicit Euler





Summary Explicit/Implicit Euler

$$u'(t) = F(t, u)$$

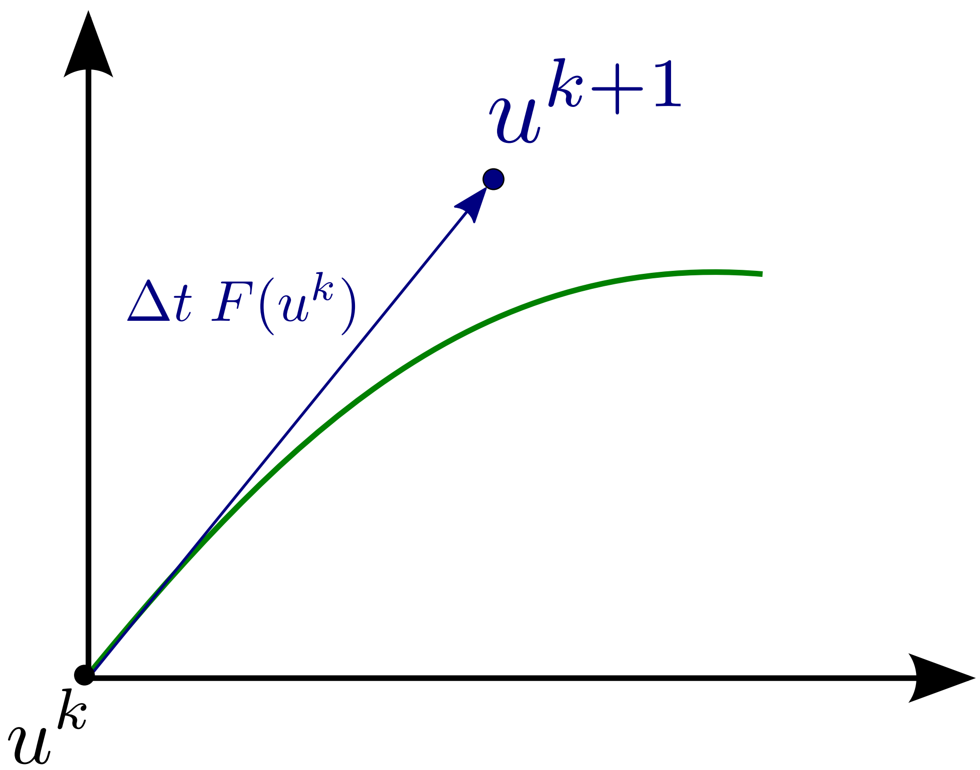
Explicit Euler: $$\frac{u^{k+1} - u^k}{\Delta t} = F(u^k)$$

Implicit Euler: $$\frac{u^{k+1} - u^k}{\Delta t} = F(u^{k+1})$$

(F must be inverted)

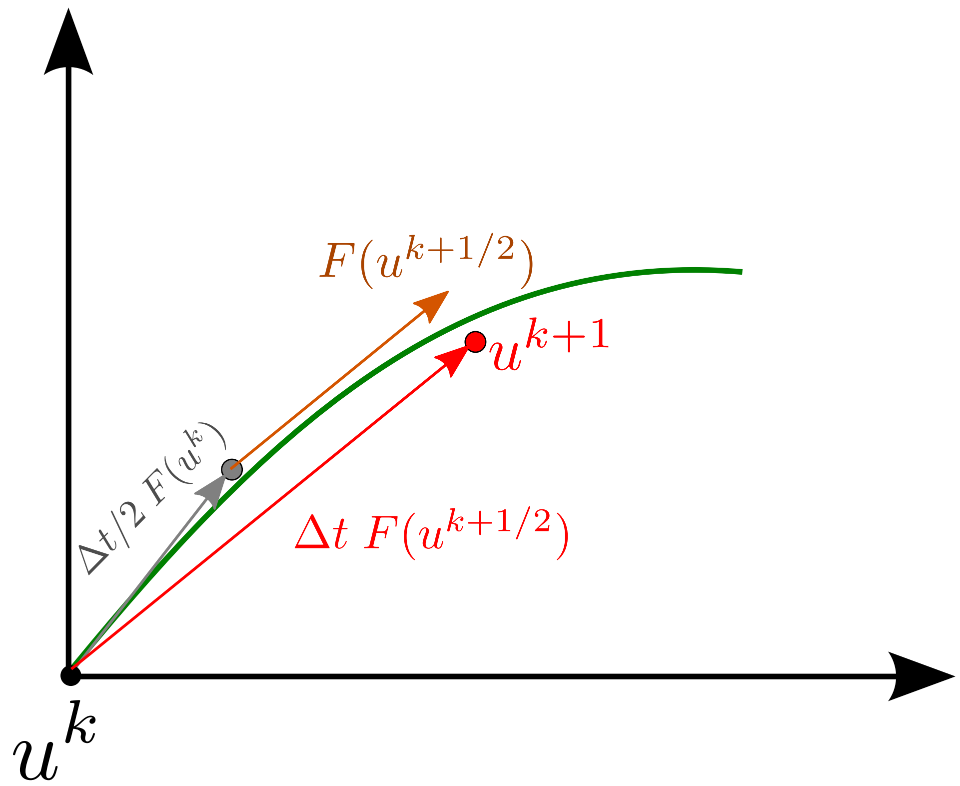
Runge Kutta

Explicit Euler



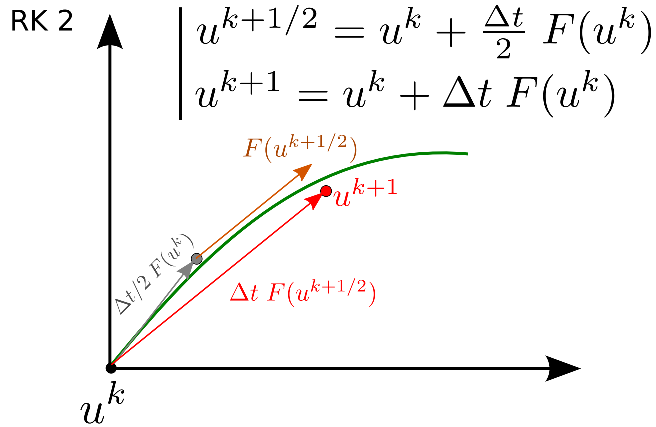
$$u^{k+1} = u^k + \Delta t\ F(u^k)$$

Runge Kutta (order 2)



$$\left| \begin{aligned} u^{k+1/2} &= u^k + \frac{\Delta t}{2}\ F(u^k) \\ u^{k+1} &= u^k + \Delta t\ F(u^{k+1/2}) \end{aligned} \right.$$

Runge Kutta: Free fall

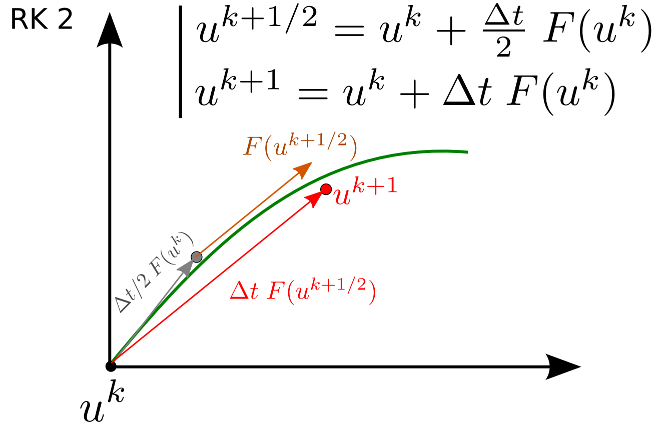


$$F : u^k = (x^k, v^k) \mapsto \begin{pmatrix} v^k \\ g \end{pmatrix}$$

$$u^{k+1/2} = \begin{pmatrix} x^k + \Delta t/2 \, v^k \\ v^k + \Delta t/2 \, g \end{pmatrix}$$

$$F(u^{k+1/2}) = \begin{pmatrix} v^k + \frac{\Delta t}{2} g \\ g \end{pmatrix}$$

Runge Kutta: Free fall



$$x^{k+1} = x^k + \Delta t\; v^k + \frac{(\Delta t)^2}{2} g$$
$$v^{k+1} = v^k + \Delta t g$$

$$\begin{cases} x^{k+2} = 2x^{k+1} - x^k + (\Delta t)^2 g \\ x^0 = x_0 \\ x^1 = x_0 + \Delta t\; v_0 + \dfrac{(\Delta t)^2}{2} g \end{cases} \Rightarrow \boxed{x^k = x_0 + (k\,\Delta t)\; v_0 + \frac{(k\Delta t)^2}{2} g}$$

Runge Kutta: ode45

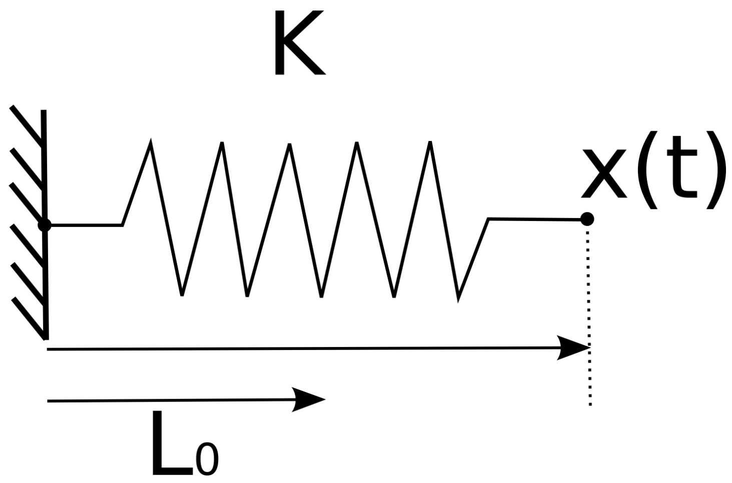
$$
\begin{aligned}
k_1 &= F(t_n, u_n) \\
k_2 &= F\left(t_n + \tfrac{1}{5}\Delta t, u_n + \tfrac{1}{5}\Delta t\ k_1\right) \\
k_3 &= F\left(t_n + \tfrac{3}{10}\Delta t, u_n + \left(\tfrac{3}{40}k_1 + \tfrac{9}{40}k_2\right)\Delta t\right) \\
k_4 &= F\left(t_n + \tfrac{3}{5}\Delta t, u_n + \left(\tfrac{3}{10}k_1 - \tfrac{9}{10}k_2 + \tfrac{6}{5}k_3\right)\Delta t\right) \\
k_5 &= F\left(t_n + \Delta t, u_n + \left(-\tfrac{11}{54}k_1 + \tfrac{5}{2}k_2 - \tfrac{70}{27}k_3 + \tfrac{35}{27}k_4\right)\right) \\
k_6 &= F\left(t_n + \tfrac{7}{8}\Delta t,\right. \\
&\quad \left. u_n + \left(\tfrac{1631}{55296}k_1 + \tfrac{175}{512}k_2 - \tfrac{575}{13824}k_3 + \tfrac{44275}{110592}k_4 + \tfrac{253}{4096}k_5\right)\Delta t\right)
\end{aligned}
$$

$$
\begin{aligned}
u_{n+1}^4 &= u_n + \Delta t\left(\tfrac{2825}{27648}k_1 + \tfrac{18575}{48384}k_3 + \tfrac{13525}{55296}k_4 + \tfrac{277}{14336}k_5 + \tfrac{1}{4}k_6\right) \\
u_{n+1}^5 &= u_n + \Delta t\left(\tfrac{37}{378}k_1 + \tfrac{250}{621}k_3 + \tfrac{125}{594}k_4 + \tfrac{512}{1771}k_6\right)
\end{aligned}
$$

Spring Mass System

Spring force

$$F(t) = K(L_0 - x(t))$$



Equation of motion

$$x''(t) = K/m(L_0 - x(t))$$

ODE formulation

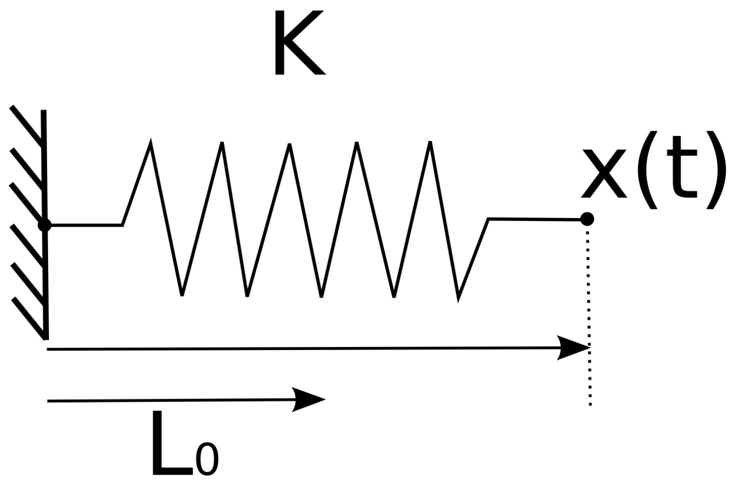
$$u'(t) = \mathrm{A}u(t) + \mathrm{B}$$

$$\mathrm{A} = \begin{pmatrix} 0 & 1 \\ -K/m & 0 \end{pmatrix} \quad \mathrm{b} = \begin{pmatrix} 0 \\ L_0\, K/m \end{pmatrix}$$

Spring Mass System

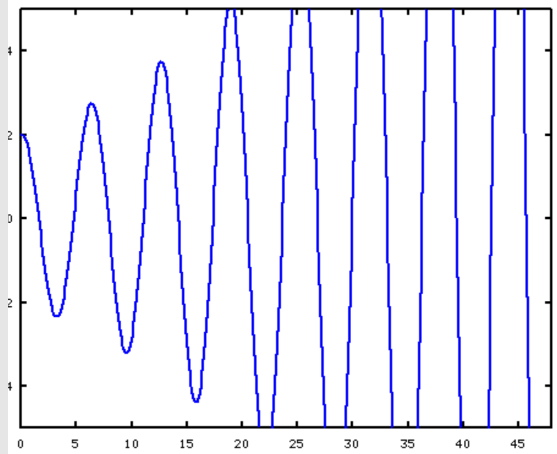
Spring force

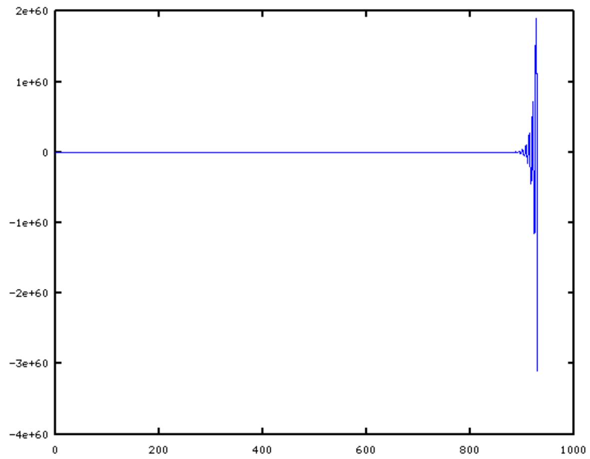
$$F(t) = K(L_0 - x(t))$$



Explicit Euler:

$$x^{k+2} = 2x^{k+1} - \left(1 + (\Delta t)^2 \frac{K}{m}\right) x^k + (\Delta t)^2 \frac{K}{m} L_0$$





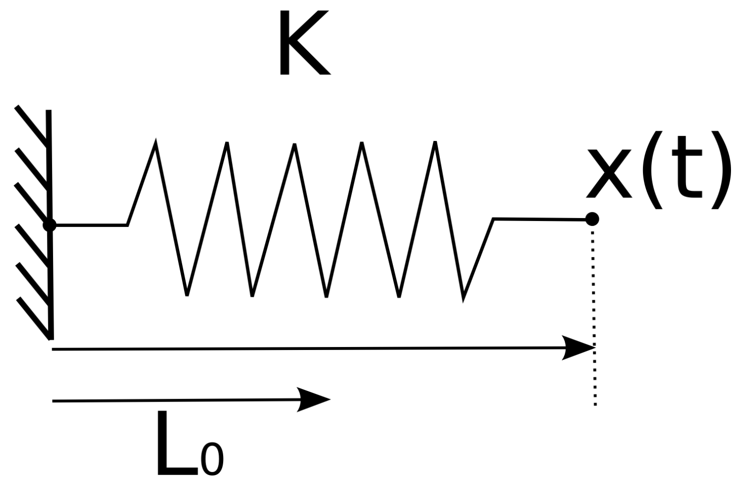
Expect:

$$x(t) = A \sin(\omega t + \varphi)$$

Spring Mass System

Spring force

$$F(t) = K(L_0 - x(t))$$



Explicit Euler:

$$x^{k+2} = 2x^{k+1} - \left(1 + (\Delta t)^2 \frac{K}{m}\right) x^k + (\Delta t)^2 \frac{K}{m} L_0$$

Do not diverge if $1 + \sqrt{(\Delta t)^2 \frac{K}{m}}) < 1$

=> Always diverge to infinity !

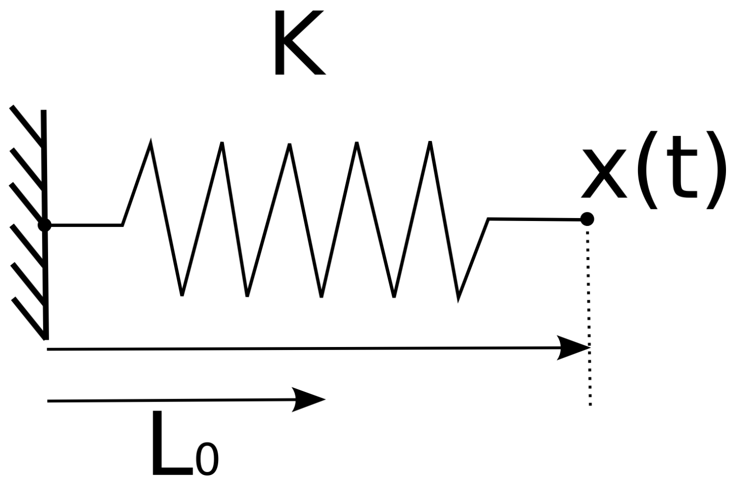
Spring Mass System

Spring force

$$F(t) = K(L_0 - x(t))$$

Add fluid damping

$$F_d(t) = -\mu \, v(t)$$



New equation for explicit Euler:

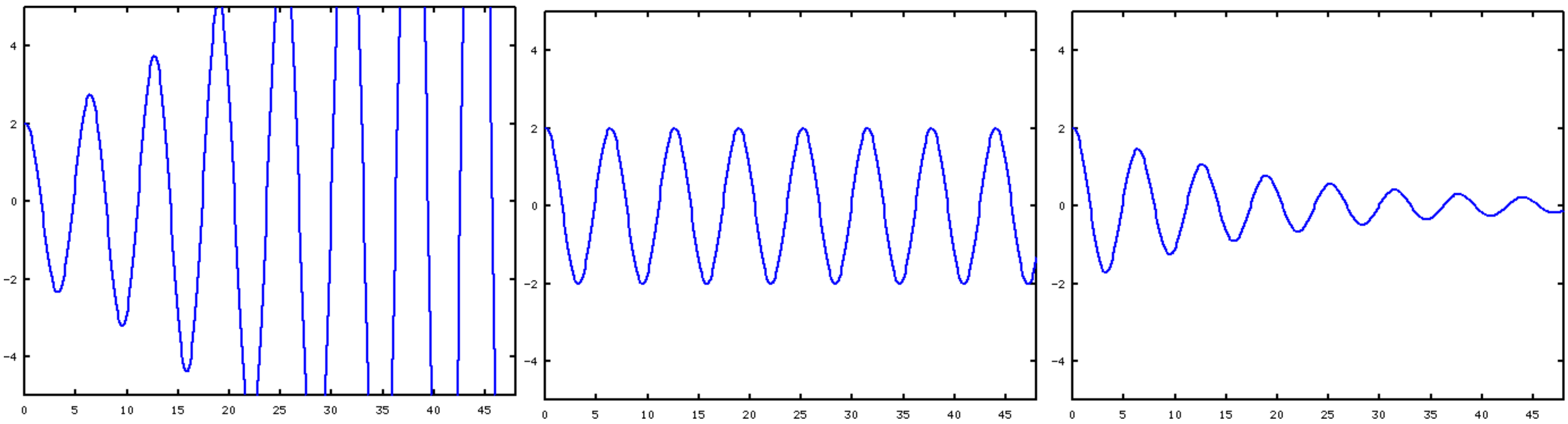
$$x^{k+2} - \left(2 - \frac{\mu}{m}\Delta t\right) x^{k+1} + \left(1 + (\Delta t)^2 \frac{K}{m} - \frac{\mu}{m}\Delta t\right) x^k = (\Delta t)^2 \frac{K}{m} L_0$$

Conditionnaly stable

Large K => Stiff springs

Stiff ODE

Accuracy != Stability



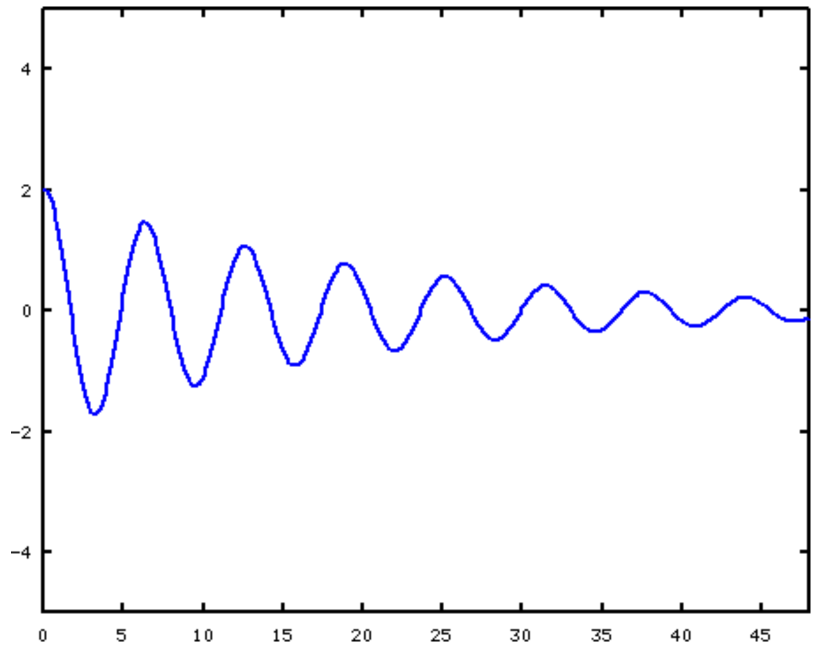
Implicit Euler

$$\begin{pmatrix} 1 & -\Delta t \\ -\Delta t \frac{K}{m} & 1 \end{pmatrix} u^{k+1} = u^k + \Delta t \begin{pmatrix} 0 \\ K \frac{L_0}{m} \end{pmatrix}$$

M

Eigenvalues of M^{-1} $\quad 1 - \Delta t \sqrt{\frac{K}{m}} < 1$

Unconditionally stable

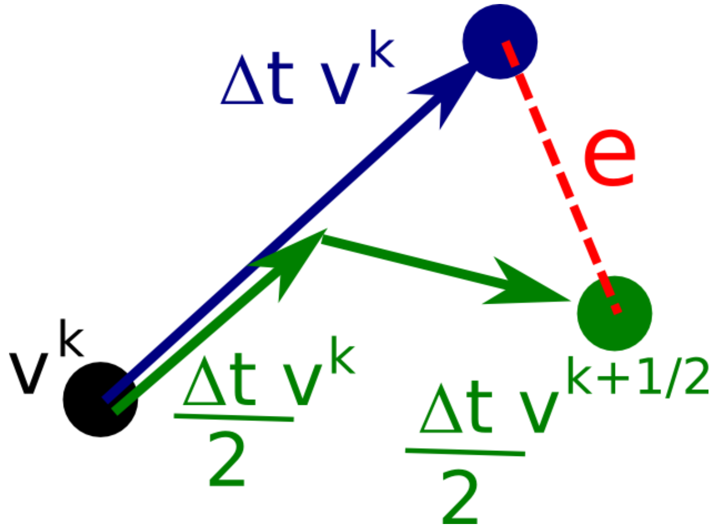


Automatic step-size

Compute:

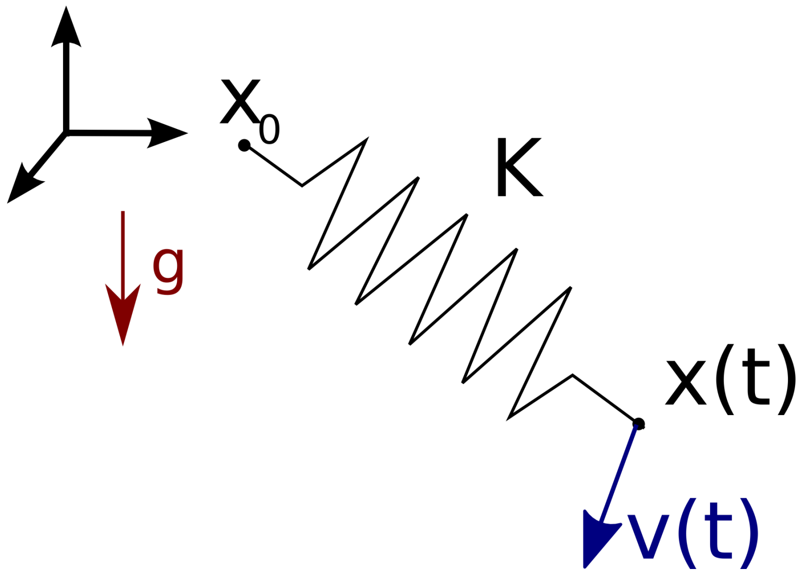
- $u_1^{k+1} = u^k + \Delta t\ F(u^k)$
- $\left| \begin{array}{l} u_2^{k+1/2} = u^k + \Delta t/2\ F(u^k) \\ u_2^{k+1} = u_2^{k+1/2} + \Delta t/2\ F(u_2^{k+1/2}) \end{array} \right.$
- $e = \|u_1^{k+1} - u_2^{k+1}\|$

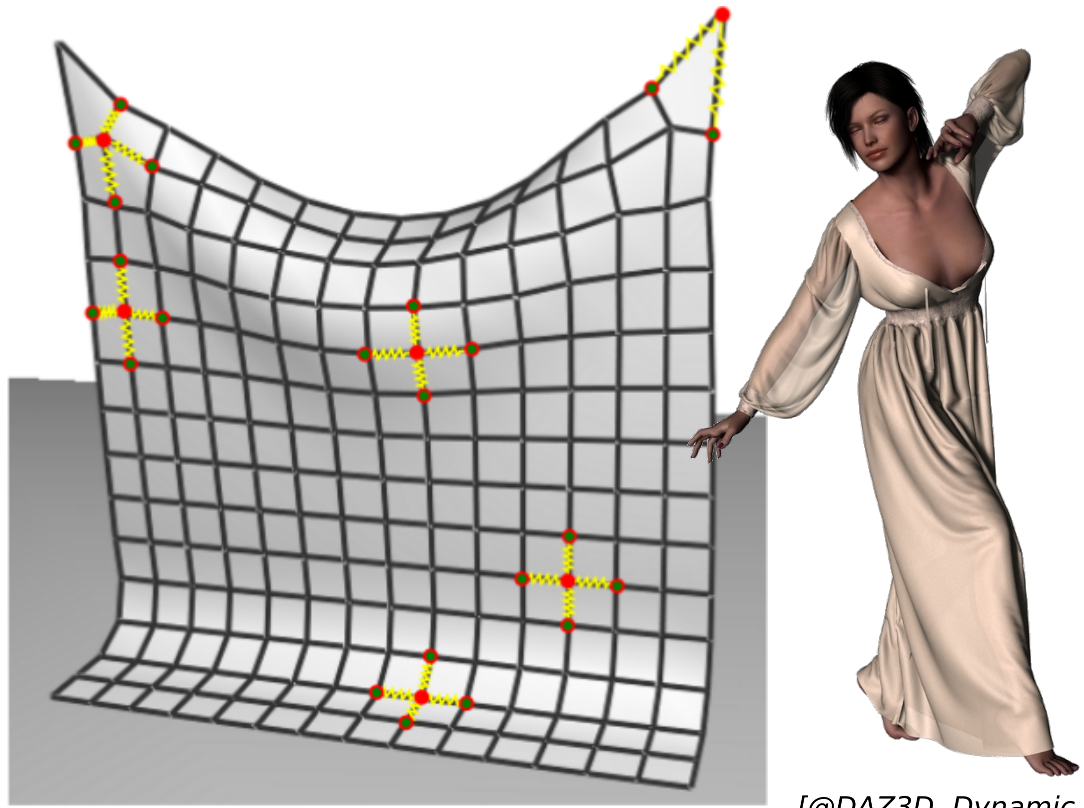
$$\left| \begin{array}{l} e < K_{\max} \Rightarrow (\Delta t)_{\text{new}} = \Delta t/2 \\ e < K_{\min} \Rightarrow (\Delta t)_{\text{new}} = 2\Delta t \end{array} \right.$$



Example of simulation

$$F(t) = K\left(L_0 - \|x - x_0\|\right)\frac{x - x_0}{\|x - x_0\|}$$





[@DAZ3D, Dynamic Clothing]