Level-Set. Application aux contours de topologie variable.

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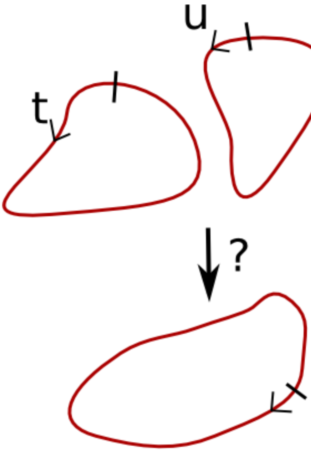
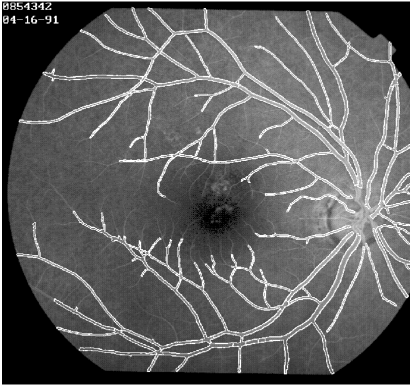
Limite des approches à topologie fixe

- Parametrage de la courbe.





Rappel : Topologie

Wikipedia

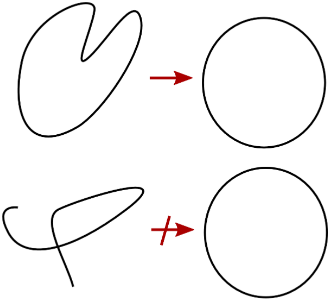
Un espace topologique est un couple (E, T), où E est un ensemble de T, et T un ensemble de parties de E tel que

1. L'ensemble vide et E appartient à T.
2. Toute réunion d'ouverts est un ouvert, c'est-à-dire si $(O_i)_{i \in I}$ est une famille de T, alors $\bigcup O_i \in T$.
3. Toute intersection finie d'ouverts est un ouvert, c'est-à-dire si $(O_i)_{i \in I}$ sont des élements de T, alors $\bigcap O_i \in T$.

Rappel : Topologie

- S_1 et S_2 sont toplogiquement équivalent (homéomorphe) si il existe un homeomorphisme de S_1 vers S_2.
- **Homéomorphisme** = Application bijective continue.

$\Rightarrow$ Traduction : *2 Courbes ont même topologie si on peut déformer l'une en l'autre sans déchirure si couture.*

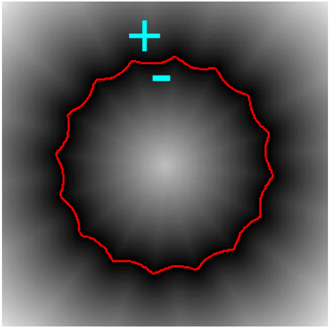


Idée

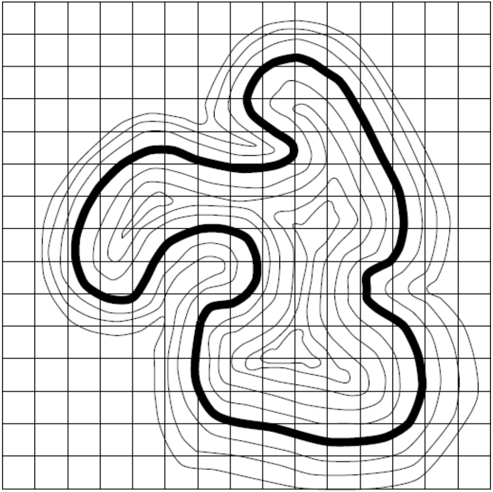
- Ne pas gérer explicitement la topologie grâce à une description implicite.

$$\begin{cases} C = \{(x,y) \in \mathbb{R}^2 | \phi(x,y) = 0\} \\ C = \phi^{-1}(0) \end{cases}$$

- $\phi : \mathbb{R}^2 \mapsto \mathbb{R}$ fonction de potentiel (field function).

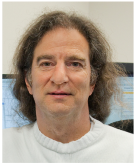


Idée



Inventeurs

- Stanley Osher (UCLA), Applied math.
- James Sethian (UC Berkeley), Applied math.
- Ronald Fedkiw (Standford), CG.



S. Osher, J. Sethian. Fronts Propagating with Curvature Dependent Speed : Algorithms Based on Hamilton-Jacobi Formulations. *Journal of Computational Physics.* 1988.

Fronts Propagating with Curvature Dependent Speed:

Algorithms Based on Hamilton-Jacobi Formulations

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We devise new numerical algorithms, called PSC algorithms, for following fronts propagating with curvature-dependent speed. The speed may be an arbitrary function of curvature, and the front can also be passively advected by an underlying flow. These algorithms approximate the equations of motion, which resemble Hamilton-Jacobi equations with parabolic right-hand-sides, by using techniques from the hyperbolic conservation laws. Non-oscillatory schemes of various orders of accuracy are used to solve the equations, providing methods that accurately capture the formation of sharp gradients and cusps in the moving fronts. The algorithms handle topological merging and breaking naturally, work in any number of space dimensions, and do not require that the moving surface be written as a function. The methods can be also used for more general Hamilton-Jacobi-type problems. We demonstrate our algorithms by computing the solution to a variety of surface motion problems.

[1] Supported by NSF Grant No., DMS85-03294, DARPA Grant in the ACMP Program, ONR Grant N00014-86-K-0691.

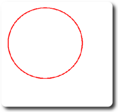
[2] Supported by the Applied Mathematics Subprogram of the Office of Energy Research under contract DE-AC03-76SF00098 and the National Science Foundation.

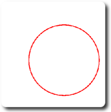
This paper appeared as

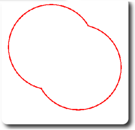
Osher, S., and Sethian, J.A., Journal of Computational Physics, 79, pp.12-49, (1988)

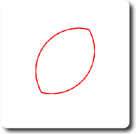
Exemples

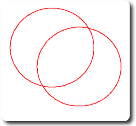
- 1 Cercle
- Union de cercles
- Intersection de cercles
- 2 Cercles





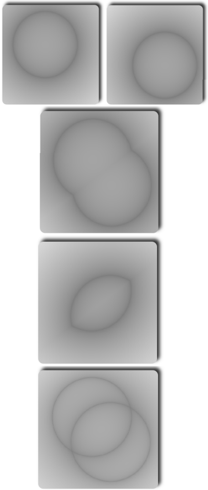






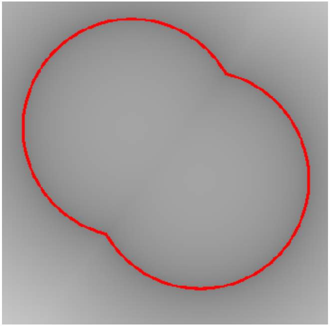
Mélange de formes

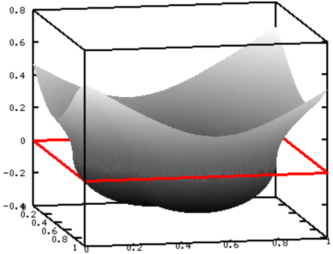
- 2 Formes de bases
  $C_1 = \phi_1^{-1}(0), C_2 = \phi_2^{-1}(0)$
  - Union
    $$C_1 \cup C_2 : \phi = \min(\phi_1, \phi_2)$$
  - Intersection
    $$C_1 \cap C_2 : \phi = \max(\phi_1, \phi_2)$$
  - Somme
    $$C_1 + C_2 : \phi = \phi_1 \phi_2$$



Déformation

- Problème : Déformer une courbe $C(u)$ en ayant connaissance du potentiel $\phi(x, y)$.





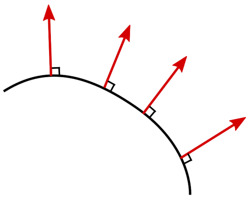
Relation courbe/potentiel

- Normales extérieures à la courbe

$$\begin{aligned}
&\forall u, \phi(C(u)) = \text{const} \\
&\Rightarrow \forall u, \frac{\partial \phi}{\partial u}(C(u)) = < \nabla\phi(C(u)), C'(u) > = 0 \\
&\Rightarrow \nabla\phi \perp C'
\end{aligned}$$

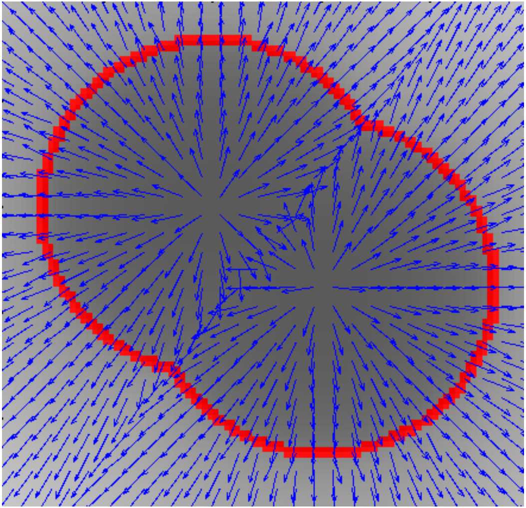
- Normale unitaire à $C(u) = (x_0, y_0)$:

$$n(x_0, y_0) = \frac{\nabla\phi(x_0, y_0)}{\|\nabla\phi(x_0, y_0)\|}$$



Analogie equipotentielles / lignes de champs

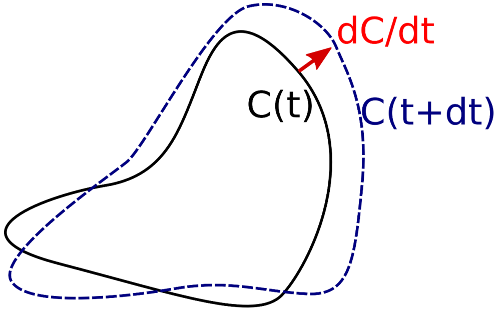
$$n = \frac{\nabla\phi}{\|\nabla\phi\|}$$



Equation de propagation

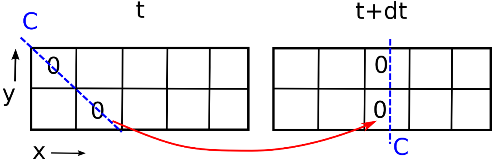
- Courbe $C(u)$ se déforme suivant vitesse $v(u)$.

$$\frac{\partial C}{\partial t} = \left(\frac{\mathrm{d}x}{\mathrm{d}t}, \frac{\mathrm{d}y}{\mathrm{d}t}\right) = v$$



Equation de propagation

- Vue *Eulerienne* sur ϕ sur l'interface $C = \phi^{-1}(0)$:



$$\phi(x + \mathrm{d}x, y + \mathrm{d}y, t + \mathrm{d}t) = \phi(x, y, t) \Rightarrow \mathrm{d}\phi(x, y, t) = 0$$

$$\begin{aligned} \mathrm{d}\phi &= \frac{\partial \phi}{\partial x}\mathrm{d}x + \frac{\partial \phi}{\partial y}\mathrm{d}y + \frac{\partial \phi}{\partial t}\mathrm{d}t = 0 \\ \Rightarrow \frac{\partial \phi}{\partial t} &= -\left(\frac{\partial \phi}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial \phi}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}t}\right) \\ \Rightarrow \phi_{,t} + &< \nabla\phi, v >= 0 \end{aligned}$$

Equation de propagation

Evolution de la courbe

$$C_{,t} = v$$

Evolution du potentiel

$$\phi_{,t} + < v, \nabla\phi >= 0$$

- Propagation equation, Convection equation.
- Avantage : ϕ indépendant de la topologie de C.

Evolution normale

- *Rem* Seule la composante normale de $v_n =< v, n >$ importe
  - Rappel : $n = \frac{\nabla\phi}{\|\nabla\phi\|}$

$$
\begin{array}{l}
\phi_{,t} + < v, \nabla\phi >= 0 \\
\Leftrightarrow \phi_{,t} + < v, n > \|\nabla\phi\| = 0 \\
\Leftrightarrow \phi_{,t} + v_n \|\nabla\phi\| = 0
\end{array}
$$

Equation d'evolution equivalente

$$\phi_{,t} + < v, \nabla\phi >= 0$$

$$\phi_{,t} + v_n \|\nabla\phi\| = 0$$

Ex. Rotation d'un cercle

Evolution suivant la normale

- Dans le cas d'une courbe C :

$$C_{,t} = a\, n$$

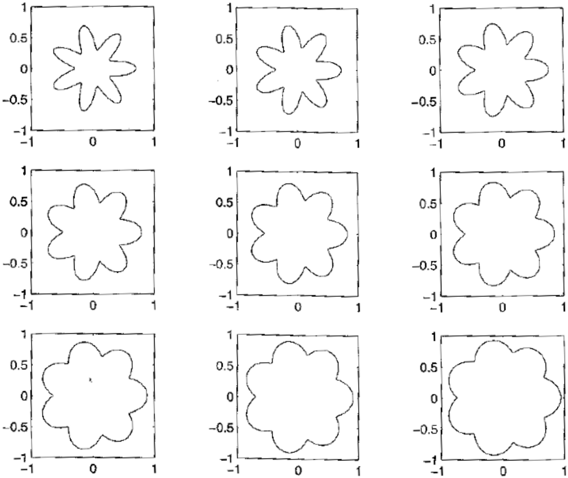
  - $a > 0$ dilatation
  - $a < 0$ contraction

- Equation sur ϕ :

$$\phi_{,t} + a\,\|\nabla\phi\| = 0$$

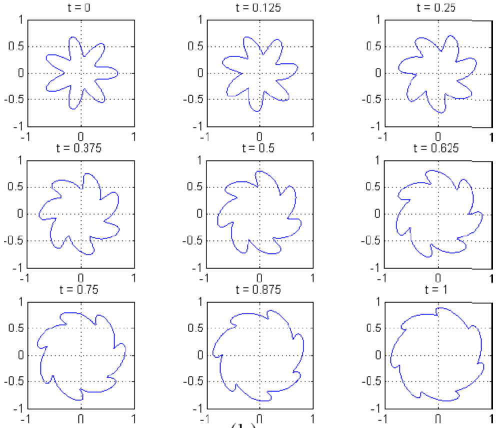
Evolution suivant la normale

Exemples



Evolution suivant la normale

Exemples : Quel est la vitesse ?



(b)

Evolution suivant la courbure

La courbure (moyenne) : $\kappa = \mathrm{div}(n)$.
Dans le cas d'une courbe C :

- Angle formés par les normales entre u et $u + \mathrm{d}u$.

$$\kappa(u) = \frac{\|C'(u) \times C''(u)\|}{\|C'(u)\|^3} = (\nabla \cdot n)(u)$$

- Calcul à partir de ϕ :

$$\kappa = \nabla \cdot \frac{\nabla \phi}{\|\nabla \phi\|}$$

Evolution suivant la courbure

Courbe se lissant dans les zones de forte courbure :

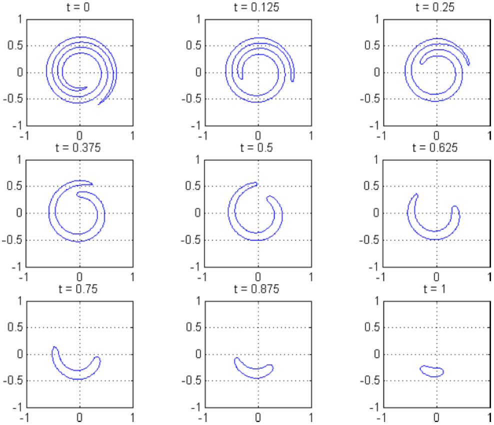
- Equation de la courbe :

$$C_{,t}(u,t) = -\kappa(u,t)n(u,t)$$

- Equation sur ϕ :

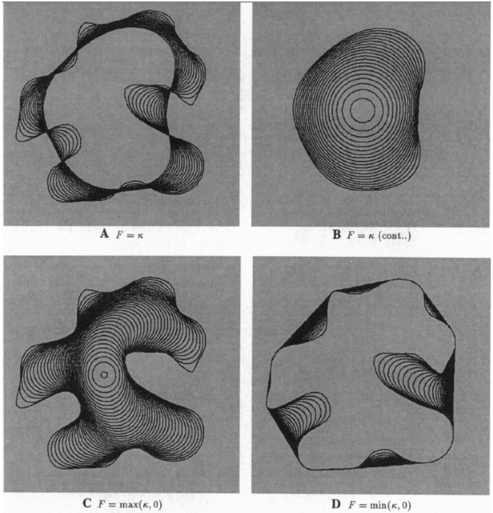
$$\phi_{,t} - \nabla \cdot \frac{\nabla\phi}{\|\nabla\phi\|}\|\nabla\phi\| = 0$$

Exemples



Exemples

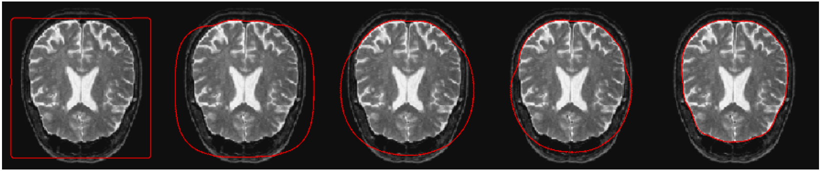
Exemples



Application a la detection de contours

Rappel des Snakes :

- On se place sur le maximum du gradient



$$\text{minimise } E(C) = \int_C \underbrace{\omega_1 \|C'\|^2 + \omega_2 \|C''\|^2}_{\text{force interne}} - \underbrace{\omega_3 \|\nabla I\|^2(C)}_{\text{force externe}}$$

Application a la detection de contours

- Force interne (extension/contraction et lissage)

$$F(\phi) = a - \kappa(\phi)$$

- Force externe

$$G(I) = 1/(1 + \|\nabla I\|)$$

- Equation d'evolution :

$$\phi_{,t} = G(I)F(\phi)\|\nabla\phi\|$$

$$\phi_{,t} = \frac{1}{1 + \|\nabla I\|}(\kappa(\phi) - a)\|\nabla\phi\|$$

Discretisation

- On veut résoudre

$$\frac{\partial \phi}{\partial t}(x,y,t) = \underbrace{\frac{1}{1+\|\nabla I(x,y)\|}}_{W_{ext}(x,y)} \underbrace{\left(\left(\nabla \cdot \frac{\nabla \phi}{\|\nabla \phi\|}\right)(x,y,t) - a\right)\|\nabla \phi(x,y,t)\|}_{W_{int}(x,y,t)}$$

- Discretisation spatiale à t fixée :

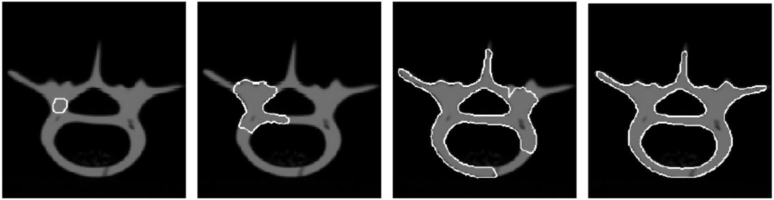
$$\frac{\partial \phi}{\partial t}(t) = W_{ext} W_{int}(t)$$

- Discrétisation temporelle :

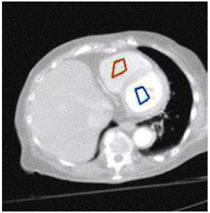
$$\phi^{t+1} = \phi^t + \Delta t \; W_{ext} W_{int}^t$$

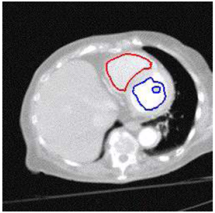
Application a la detection de contours

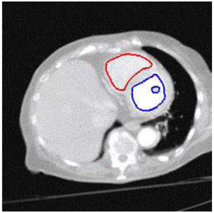
Exemples



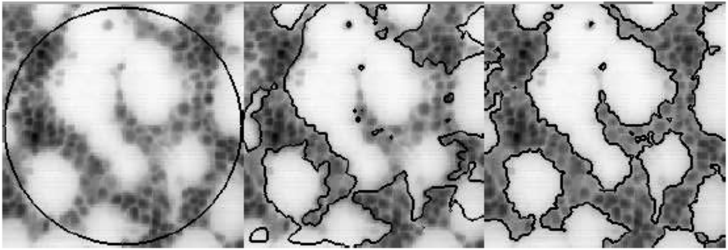
Application a la detection de contours





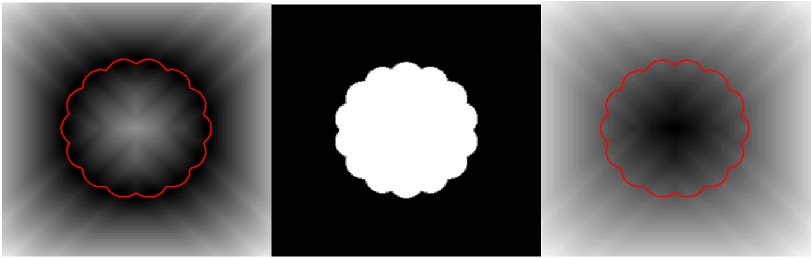


Application a la detection de contours



Annexe : Construction d'une fonction de distance signée

1. Fonction de distance non signée : (ex. Danielson)
2. Ajout du signe par propagation.



Implementation

```
obj<-image du contour
dist=Danielson(obj);
sign=propagation(obj);

phi=dist.*sign;

for k=0:N
  [phi_x,phi_y]=gradient(phi);
  vn=calcul_vitesse_normal(image,phi);
  phi = phi + dt*(vn.*(phi_x^2+phi_y^2)^(0.5));
end
```

Limitations

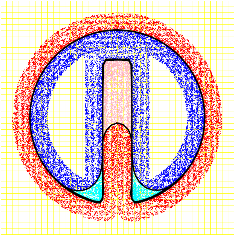
- Problème de **divergence** : Methode de resolutions.
- **Interpolation** : perte de matière.
- **Cout de calcul** : Réinitialisation de la fonction de distance.

Divergence

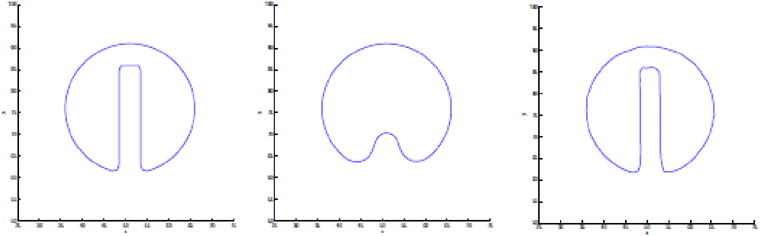
PDE : non linéaire, non lisse.

- Traitement séparé de la discrétisation gauche-droite de droite-gache.
  $\Rightarrow$ **ENO (Essentially Non Oscillatory) scheme.**
- ou : **TVD-Runge Kutta scheme**.
- Réinitialisation régulière de la fonction de distance.

Interpolation



- Ajout d’une contrainte de conservation d’aire(/volume).
- **Semi-Lagrangian particle level set**



Fast marching

- Réinitialisation de la fonction de distance : Re-calcul local : *Fast marching method*.

J. Sethian. A Fast Marching Level Set Method for Monotonically Advancing Fronts. *Proceedings of National Academy of Sciences.* 1996.

A Fast Marching Level Set Method for Monotonically Advancing Fronts *

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Dept. of Mathematics
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Oct. 20, 1995
Submitted for Publication:
Proceedings of the National Academy of Sciences

Abstract

We present a fast marching level set method for monotonically advancing fronts, which leads to an extremely fast scheme for solving the Eikonal equation. Level set methods are numerical techniques for computing the position of propagating fronts. They rely on an initial value partial differential equation for a propagating level set function, and use techniques borrowed from hyperbolic conservation laws. Topological changes, corner and cusp development, and accurate determination of geometric properties such as curvature and normal direction are naturally obtained in this setting. In this paper, we describe a particular case of such methods for interfaces whose speed depends only on local position. The technique works by coupling work on entropy conditions for interface motion, the theory of viscosity solutions for Hamilton-Jacobi equations and fast adaptive narrow band level set methods. The technique is applicable to a variety of problems, including shape-from-shading problems, lithographic development calculations in microchip manufacturing, and arrival time problems in control theory.

1 Introduction

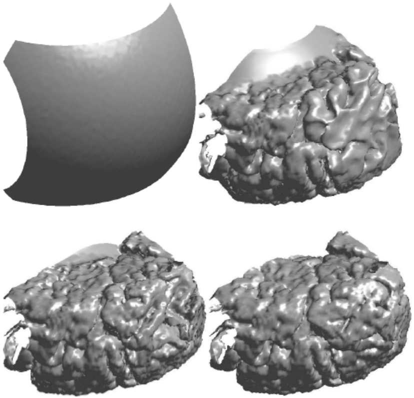
In this paper, we describe and test a numerical algorithm for tracking the evolution of interfaces. The technique applies in the case of a front propagating normal to itself with a speed F that

*Supported in part by the Applied Mathematics Subprogram of the Office of Energy Research under DE-AC03-76SF00098, and the National Science Foundation DARPA under grant DMS-8919074.

1

Extension : 3D

- Extension à la 3D immédiate
  - couteux
  - mémoire



Extension : Forces controlées par phénomènes physiques

Extension : Forces controlées par phénomènes physiques