A Bloch Torrey Equation for Diffusion in a Deforming Media

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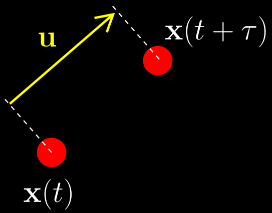
Diffusion Process

- Link Between **Microscopical** and **Macroscopical** Behavior.
- Expressed with the **Diffusion Coefficient**
  - Scalar Case:

$$6\tau D = [x(t+\tau) - x(t)]^2$$

  - Vectorial Case:

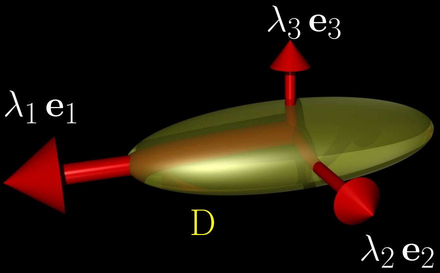
$$\begin{cases} 6\tau \mathrm{D} = \mathbf{u}\mathbf{u}^T \\ \mathbf{u} = \mathbf{x}(t+\tau) - \mathbf{x}(t) \end{cases}$$



The Diffusion Tensor

- D is a **Symetric Definite Positive** matrix by definition.

$$\mathrm{D} = \sum_{i=1}^{3} \lambda^i \mathbf{e}_i \mathbf{e}_i^T = \mathrm{R} \Lambda \mathrm{R}^T$$



The Diffusion Equation

- For a scalar ϕ

$$\frac{\partial \phi}{\partial t} = \nabla \cdot \underbrace{(\mathrm{D} \nabla \phi)}_{\text{flux density}}$$

- For a vector $\boldsymbol{\phi} = \phi^i \, \mathbf{e}_i$

$$\frac{\partial \phi^i}{\partial t} = \nabla \cdot \left(\mathrm{D} \nabla \phi^i\right)$$

- General Solution (D independant of t with boundary conditions sent to infinity.)

$$\phi(\mathbf{x}, t) = \frac{1}{\sqrt{|\mathrm{D}|}\,(4\pi\, t)^{\frac{N}{2}}} e^{-\frac{\mathbf{x}^T\, \mathrm{D}^{-1}\, \mathbf{x}}{4\, t}} \ * \ \phi(\mathbf{x}, 0)$$

Illustration of the Diffusion Process

Exemple of the Action of the **Orientation** of the Diffusion Tensor:

1. Original Distribution
2. Filtered Distribution
3. Main Orientation of the Tensors

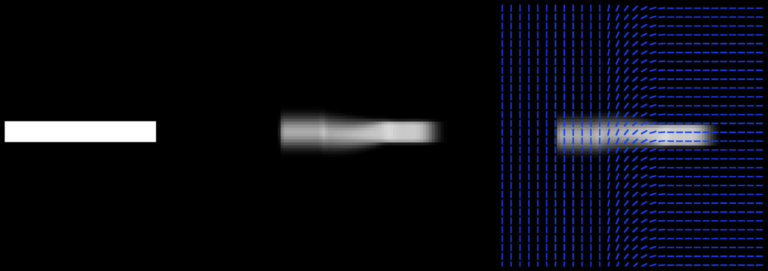


Illustration of the Diffusion Process (II)

Exemple of the Action of the **Inhomogeneous** Diffusion Phenomena Applied to the Filtering.

1. Original
2. Noisy
3. Homogeneous Gaussian Filtering
4. Inhomogeneous Diffusion

Bloch Equation

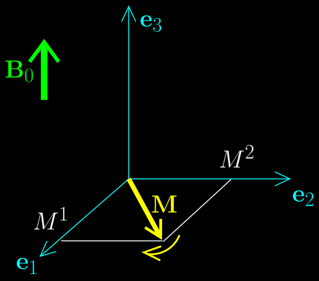
- $^1\mathrm{H}$ atoms abundant in the water possess a nuclear angular momentum: the **Spin**.
- The orientation of the Spin is given by $\mathbf{M}$.
- Under a Magnetic Field $\mathbf{B}$, the momentum rotates around $\mathbf{B}$ at the pulsation $\gamma\|\mathbf{B}\|$:

$$\mathbf{M}_{,t} = \mathbf{M} \times \gamma\mathbf{B}$$

Bloch Equation (II)

- In order to acquire the momentum $\mathbf{M}$, a large Magnetic Field $\mathbf{B}_0$ is applied along the axis $z : \mathbf{e}_3$, and $\mathbf{M}$ is flipped in the (x, y) plane by a special field.

$$\begin{cases} \mathbf{M}_{,t} = \mathbf{M} \times \gamma \mathbf{B} - \frac{M^1 \mathbf{e}_1 + M^2 \mathbf{e}_2}{T_2} - \frac{M^3 - M^3_0}{T_1} \mathbf{e}_3 \\ \mathbf{M}(\mathbf{x}, 0) = \mathbf{M}_0 \end{cases}$$



Bloch-Torrey Equation

The Diffusive term $\nabla \cdot (\mathrm{D}\nabla \quad)$ is added:

$$\mathbf{M}_{,t} = \mathbf{M} \times \gamma \mathbf{B} - \frac{M^1 \mathbf{e}_1 + M^2 \mathbf{e}_2}{T_2} - \frac{M^3 - M_0^3}{T_1} \mathbf{e}_3 + \nabla \cdot (D \, \nabla \mathbf{M})$$

Where $\nabla \cdot (\mathrm{D} \, \nabla \mathbf{M})$ has to be understood componentwise.

Attenuation Expression

It is first supposed that

- D does not depends on t, then for every position $\mathrm{D} = \mathit{const}$.
- The diffusion seen by each molecule is constant along its displacement.

Attenuation Expression

- Only the (x, y) Components are Tacken in Account:

$$\underline{M} = M^1 + i\, M^2$$

- The Magnetization Vector is Expressed as:

$$\underline{M}(\mathbf{x}, t) = A_{\mathbf{x}}(t)\, e^{-\alpha(t)}\, e^{i\varphi(\mathbf{x},t)}$$

- The matrix B is defined:

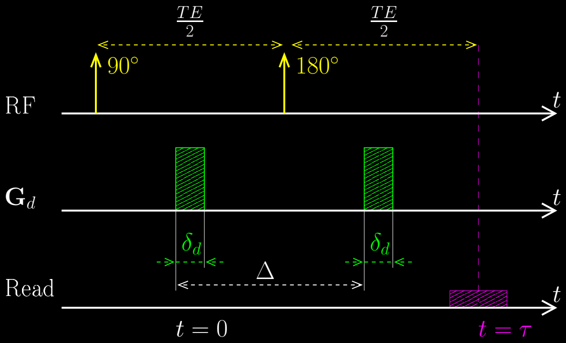
$$\begin{cases} \mathrm{B}(\mathbf{x}, t) = (\nabla\varphi)\,(\nabla\varphi)^T \\ \varphi = \gamma \int_0^t \mathbf{x} \cdot \mathbf{G}(t')\, \mathrm{d}t' \quad !!! \end{cases}$$

- The Attenuation $A_{\mathbf{x}}$ is given by:

$$\ln\left(\frac{A_{\mathbf{x}}(t)}{A_{\mathbf{x}}(0)}\right) = -\mathrm{tr}\left[\left(\int_0^t \mathrm{B}(\mathbf{x}, t')\, \mathrm{d}t'\right)\, \mathrm{D}\right]$$

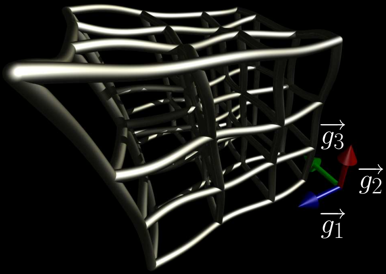
Special Pulse Sequence

$$\begin{cases} \ln\left(\frac{A_x(t)}{A_x(0)}\right) = -\Delta\, \mathbf{k}^T \mathrm{D} \mathbf{k} \\ \mathbf{k} = \gamma\, \delta\, \mathbf{G}_d \end{cases}$$



Now the material is dynamic

- The position $\mathbf{x}$ is depending on the time.
- Use of an original **Underformed Referential** given by $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ and $\mathbf{X} = X^i \mathbf{e}_i$.
- Addition of a **Deformed Referential** using the **Curvilinear Coordinate** system given by $(\mathbf{g}_1, \mathbf{g}_2, \mathbf{g}_3)$ and $\xi = \xi(\mathbf{X}, t)$.



Deformed Referential

- The deformation is characterized by the tensorial **Deformation Gradient**:

$$\mathrm{F} = \frac{\partial \xi^i}{\partial X^j}$$

- And follow the relation:

$$\mathrm{d}\boldsymbol{\xi} = \mathrm{F}\ \mathrm{d}\mathbf{X}$$

Expression of the gradient of phase

- The spatial phase variation has to be expressed in the fixed referential where the phase is:

$$\varphi(\xi, t) = \gamma \int_0^t \mathbf{X}(\xi, t') \cdot \mathbf{G}(t') \, \mathrm{d}t'$$

- It is assumed a smooth deformation:

$$\nabla^T \varphi(\xi, t) \, \mathrm{d}\xi = \nabla^T \varphi(\mathbf{X}, t) \, \mathrm{d}\mathbf{X}$$

- Using the deformation Gradient F:

$$\nabla \varphi(\xi, t) = \mathrm{F}^{-T}(\mathbf{X}, t) \, \nabla \varphi(\mathbf{X}, t) \, \mathrm{d}\mathbf{X}$$

Expression of the Diffusion tensor

The component of the tensor depends on the basis:

- The tensor expressed in the original referential: $\overline{D}$
- The tensor expressed in the deformed referential: D
- They are linked by the relation:

$$\mathrm{D}_i^{\,j} = \frac{\partial \xi^i}{\partial X^k} \frac{\partial X^l}{\partial \xi^j} \overline{\mathrm{D}}_l^{\,k}$$

$$\Rightarrow \mathrm{D} = \mathrm{F}\,\overline{\mathrm{D}}\,\mathrm{F}^{-1}$$

Expression of the Attenuation

- The Attenuation is Expressed with the Components of the Initial Referential:

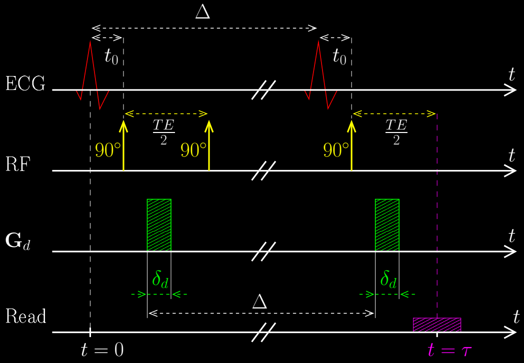
$$\ln\left(\frac{A\mathbf{x}(t)}{A\mathbf{x}(0)}\right) = \int_0^t (\nabla\varphi)^T\ \overline{\mathrm{D}}\ \mathrm{F}^{-2}\,\nabla\varphi\,\mathrm{d}t'$$

- The **Right Stretch tensor** is introduced such that: $\mathrm{F}^T\,\mathrm{F} = \mathrm{U}^2$

$$\ln\left(\frac{A\mathbf{x}}{A\mathbf{x}}\right) = \int_0^t (\nabla\varphi)^T\ \overline{\mathrm{D}}\ \mathrm{U}^{-2}\,\nabla\varphi\,\mathrm{d}t'$$

Aquisition Sequence

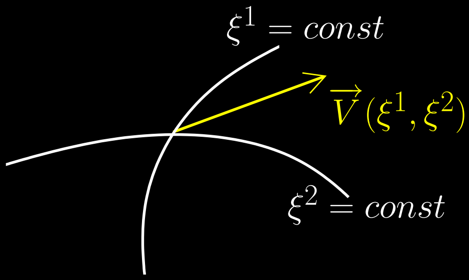
$$\begin{cases} \ln\left(\frac{A_{\mathbf{x}}(\tau)}{A_{\mathbf{x}}(0)}\right) = -\Delta\, \mathbf{k}^T\, \mathrm{D}_{\text{obs}}\, \mathbf{k} \\ \mathrm{D}_{\text{obs}} = \frac{1}{\Delta}\int_0^{\Delta} \overline{\mathrm{D}}\, \mathrm{U}^{-2}\, \mathrm{d}t \end{cases}$$



Use of the Curvilinear Coordinates

- A change of coordinates:

$$(\xi^1, \xi^2, \xi^3) = \phi(x^1, x^2, x^3)$$



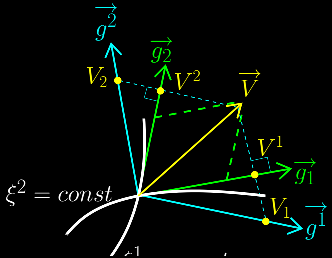
Curvilinear Basis

- A covariant basis $\mathbf{g}_i$ such that $\mathbf{x} = x^i\,\mathbf{e}_i = \xi^i\,\mathbf{g}_i$:

$$\mathbf{g}_i = \frac{\partial x^j}{\partial \xi^i}\,\mathbf{e}_j$$

- A contravariant basis $\mathbf{g}^i$:

$$\mathbf{g}^i = \frac{\partial \xi^i}{\partial x^j}\,\mathbf{e}^j$$



Parameters of the Curvilinear Coordinates

- The Metric tensor:

$$g_{ij} = \mathbf{g}_i \cdot \mathbf{g}_j$$

- The ∇ operator given by:

$$\nabla = \mathbf{g}^i \frac{\partial}{\partial \xi^i}$$

- The Christoffel symbols of second kind Γ:

$$\begin{cases} \mathbf{g}_{i,j} = \Gamma^k_{ij} \mathbf{g}_k \\ \Gamma^i_{jk} = \frac{\partial^2 x^l}{\partial \xi^j \partial \xi^k} \frac{\partial \xi^i}{\partial x^l} \end{cases}$$

Expression of the Bloch-Torrey Equation

- We use:

$$\mathbf{M}(\xi, t) = M^i(\xi, t)\, \mathbf{e}_i$$

- In the Cartesian case the Equation is:

$$\mathbf{M}_{,t} = \mathbf{M} \times \gamma \mathbf{B} - \frac{M^1 \mathbf{e}_1 + M^2 \mathbf{e}_2}{T_2} - \frac{M^3 - M_0^3}{T_1} \mathbf{e}_3 + \nabla \cdot \left(\mathrm{D}\, \nabla \mathbf{M} \right)$$

- Only the diffusion $\nabla \cdot \left(\mathrm{D}\, \nabla\, M^i \right)$ term is modified:

Expression of the Diffusion in Curvilinear Coordinates

- The first term:

$$\mathrm{D}\,\nabla M^i = \mathrm{D}_j{}^k\, M^i_{,k}\, \mathbf{g}^j$$

- The diffusion term:

$$\nabla \cdot (\mathrm{D}\,\nabla M^i) = \left[\left(\mathrm{D}_k{}^l\, M^i_{,l} \right)_{,j} - \Gamma^m{}_{kj}\, \mathrm{D}_m{}^l\, M^i_{,l} \right] \mathrm{g}^{jk}$$

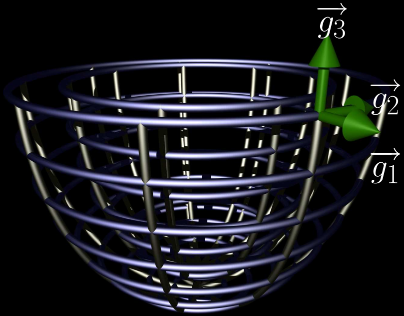
- The complete equation:

$$\mathbf{M}_{,t} = \mathbf{M} \times \gamma \mathbf{B} - \frac{M^1 \mathbf{e}_1 + M^2 \mathbf{e}_2}{T_2} - \frac{M^3 - M^3_0}{T_1} \mathbf{e}_3 + \left[\left(\mathrm{D}_j{}^k\, \mathbf{M}_{,k} \right)_{,i} - \Gamma^l{}_{ji}\, \mathrm{D}_l{}^k\, \mathbf{M}_{,k} \right] \mathrm{g}^{ij}$$

Definition of the Coordinates

- Change of variable:

$$\begin{cases} x^1 &= C \sinh(\xi^1)\sin(\xi^2)\cos(\xi^3) \\ x^2 &= C \sinh(\xi^1)\sin(\xi^2)\sin(\xi^3) \\ x^3 &= C \cosh(\xi^1)\cos(\xi^2) \end{cases}$$



Basis

- Contravariant basis vector:

$$\mathbf{g}_1 = C \begin{pmatrix} \cosh(\xi^1)\sin(\xi^2)\cos(\xi^3) \\ \cosh(\xi^1)\sin(\xi^2)\sin(\xi^3) \\ \sinh(\xi^1)\cos(\xi^2) \end{pmatrix}$$

$$\mathbf{g}_2 = C \begin{pmatrix} \sinh(\xi^1)\cos(\xi^2)\cos(\xi^3) \\ \sinh(\xi^1)\cos(\xi^2)\sin(\xi^3) \\ -\cosh(\xi^1)\sin(\xi^2) \end{pmatrix}$$

$$\mathbf{g}_3 = C \begin{pmatrix} \sinh(\xi^1)\sin(\xi^2)\sin(\xi^3) \\ -\sinh(\xi^1)\sin(\xi^2)\cos(\xi^3) \\ 0 \end{pmatrix}$$

Metric Tensor

- The Prolate basis is orthogonale.
- The metric tensor is diagonale:

$$[\mathbf{g}_{ij}] = C^2 \begin{pmatrix} \sinh^2(\xi^1) + \sinh^2(\xi^2) & 0 & 0 \\ 0 & \sinh^2(\xi^1) + \sinh^2(\xi^2) & 0 \\ 0 & 0 & \sinh(\xi^1)\sinh(\xi^2) \end{pmatrix}$$

- The element of volume is:

$$\mathrm{d}x\mathrm{d}y\mathrm{d}z = C^3 \sinh(\xi^1)\sin(\xi^2)\left(\sinh^2(\xi^1) + \sinh^2(\xi^2)\right)\mathrm{d}\xi^1\,\mathrm{d}\xi^2\,\mathrm{d}\xi^3$$

Christoffel Symbols

$$[\Gamma^1{}_{ij}] = \begin{pmatrix} \frac{\cosh(\xi^1)\sinh(\xi^1)}{\sinh^2(\xi^1)+\sin^2(\xi^2)} & \frac{\cos(\xi^2)\sinh(\xi^2)}{\sinh^2(\xi^1)+\sin^2(\xi^2)} & 0 \\ \frac{\cos(\xi^2)\sin(\xi^2)}{\sinh^2(\xi^1)+\sin^2(\xi^2)} & -\frac{\cosh(\xi^1)\sinh(\xi^1)}{\sinh^2(\xi^1)+\sin^2(\xi^2)} & 0 \\ 0 & 0 & -\frac{\sinh(\xi^1)\cosh(\xi^1)\sin^2(\xi^2)}{\sinh^2(\xi^1)+\sin^2(\xi^2)} \end{pmatrix}$$

$$[\Gamma^2{}_{ij}] = \begin{pmatrix} -\frac{\cos(\xi^2)\sin(\xi^2)}{\sinh^2(\xi^1)+\sin^2(\xi^2)} & \frac{\cosh(\xi^1)\sinh(\xi^1)}{\sinh^2(\xi^1)+\sin^2(\xi^2)} & 0 \\ \frac{\cosh(\xi^1)\sinh(\xi^1)}{\sinh^2(\xi^1)+\sin^2(\xi^2)} & \frac{\cos(\xi^2)\sin(\xi^2)}{\sinh^2(\xi^1)+\sin^2(\xi^2)} & 0 \\ 0 & 0 & -\frac{\sinh^2(\xi^1)\cos(\xi^2)\sin(\xi^2)}{\sinh^2(\xi^1)+\sin^2(\xi^2)} \end{pmatrix}$$

$$[\Gamma^3{}_{ij}] = \begin{pmatrix} 0 & 0 & \mathrm{cotanh}(\xi^1) \\ 0 & 0 & \mathrm{cotan}(\xi^2) \\ \mathrm{cotanh}(\xi^1) & \mathrm{cotan}(\xi^2) & 0 \end{pmatrix}$$

Expression of the Equation

- The equation is simplified by the use of an orthogonale coordinate system:

$$\mathbf{M}_{,t} = \mathbf{M} \times \gamma \mathbf{B} - \frac{M^1 \mathbf{e}_1 + M^2 \mathbf{e}_2}{T_2} - \frac{M^3 - M^3_0}{T_1} \mathbf{e}_3 +$$

$$+ \sum_{i=1}^{3} \left(\mathrm{g}^{ii} \left[\sum_{j=1}^{3} \left(\mathrm{D}_i{}^j \mathbf{M}_{,ji} + \mathrm{D}_i{}^j{}_{,i} \mathbf{M}_{,j} - \Gamma^j{}_{ii} \sum_{k=1}^{3} \left(\mathrm{D}_j{}^k \mathbf{M}_{,k} \right) \right) \right] \right)$$

- Possibility of animating it: For instance, an easy dilatation

$$\xi'^1 = a(t)\, \xi^1$$

Finite Differences methods in one dimension

- The central difference operator:

$$\mathscr{D}^1 M = M(x+\Delta x) - M(x-\Delta x)$$

- Second order accurate:

$$\left| \frac{\mathscr{D}^1 M}{2\Delta x} - \frac{\partial M}{\partial x} \right| \leq A \left|\Delta x\right|^2$$

- The second spatial derivative

$$\mathscr{D}^{11} M = M(x+\Delta x) - 2M(x) + M(x-\Delta x)$$

- Second order accurate too

$$\left| \frac{\mathscr{D}^{11} M}{(\Delta x)^2} - \frac{\partial^2 M}{\partial x^2} \right| = \mathcal{O}(\left|\Delta x\right|^2)$$

Vector and matrix notation

- The function is discretized on a spatial grid of N intervals of size Δx.
- The function is stored as a vector $\mathbf{u}$ such that:

$$M(k\Delta x) = u[k]$$

- The difference operator acting on the vector is the matrix:

$$[\mathscr{D}^1]_{i,j} = \begin{cases} 1 & \text{if } \ i = j-1 \\ -1 & \text{if } \ i = j+1 \\ 0 & \text{otherwise} \end{cases}$$

$$[\mathscr{D}^{11}]_{i,j} = \begin{cases} 1 & \text{if } \ i = j-1 \\ -2 & \text{if } \ i = j \\ 1 & \text{if } \ i = j+1 \\ 0 & \text{otherwise} \end{cases}$$

Solution of the Diffusion Equation

- The one dimensional diffusion equation is:

$$\frac{\partial M}{\partial t} = D \triangle M$$

- Using the notation $\mathbf{u}(t) = \mathbf{u}$ and $\mathbf{u}(t + \Delta t) = \mathbf{u}^+$. The new equation can be:

$$\frac{\mathbf{u}^+ - \mathbf{u}}{\Delta t} = \frac{D}{(\Delta x)^2} \mathscr{D}^{11} \mathbf{u} \quad \text{or} \quad \frac{\mathbf{u}^+ - \mathbf{u}}{\Delta t} = \frac{D}{(\Delta x)^2} \mathscr{D}^{11} \mathbf{u}^+$$

Explicit and Implicit method

- Two algorithms:
  - Explicit Method (easy to implement):

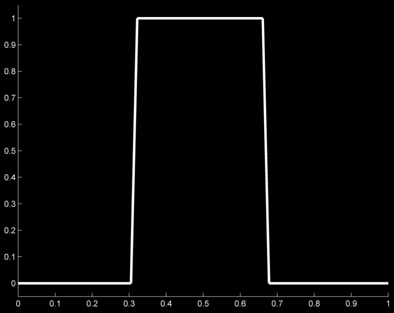
$$\mathbf{u}^{+} = \left(\mathrm{I}_N + D \frac{\Delta t}{(\Delta x)^2} \mathscr{D}^{11} \right) \mathbf{u}$$

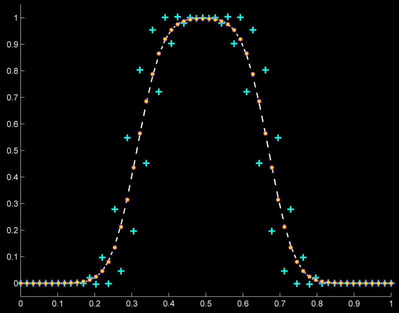
  - Implicit Method (linear system to solve):

$$\left(\mathrm{I}_N - D \frac{\Delta t}{(\Delta x)^2} \mathscr{D}^{11} \right) \mathbf{u}^{+} = \mathbf{u}$$

Comparison of the methods

- 60 samples, $\Delta x = 0.0167$, $\Delta t = 0.00149$, $D = 0.1$ and stop after 10 iterations.





Crank-Nichols Scheme

- A stable second order accurate method can also be used and states that:

$$\frac{1}{\Delta t}\mathscr{D}^{t}\mathbf{u} = \frac{D}{(\Delta x)^2}\mathscr{D}^{11}\frac{\mathbf{u}^{+} + \mathbf{u}}{2}$$

Operator in three dimensions

- The operators are defined in three dimensions:

$$\mathscr{D}^i\,\mathbf{M} = \mathbf{M}(\xi^i + \Delta\xi^i) - \mathbf{M}(\xi^i - \Delta\xi^i)$$

$$\mathscr{D}^{ii}\,\mathbf{M} = \mathbf{M}(\xi^i + \Delta\xi^i) - 2\mathbf{M}(\xi^i) + \mathbf{M}(\xi^i - \Delta\xi^i)$$

$$\begin{aligned}\mathscr{D}^{ij}\,\mathbf{M} = \quad & +\mathbf{M}(\xi^i + \Delta\xi^i, \xi^j + \Delta\xi^j)\\ & -\mathbf{M}(\xi^i - \Delta\xi^i, \xi^j + \Delta\xi^j)\\ & -\mathbf{M}(\xi^i + \Delta\xi^i, \xi^j - \Delta\xi^j)\\ & +\mathbf{M}(\xi^i - \Delta\xi^i, \xi^j - \Delta\xi^j)\end{aligned}$$

Vector Notation

- The spatial grid of size $N_1 \times N_2 \times N_3$ is created and each voxel has a volume $\Delta\xi^1 \times \Delta\xi^2 \times \Delta\xi^3$.
- The discrete function $\mathbf{M}$ is stored in a large vector $\mathbf{u}$ such that:

$$M^i(k_1\Delta\xi^1, k_2\Delta\xi^2, k_3\Delta\xi^3) = u\left[i + 3\left(k_1 + N_1\left(k_2 + N_2\, k_3\right)\right)\right]$$

Matrix Notation

- The derivative matrix are very large matrix of size $(3\,N_1\,N_2\,N_3)^2$
- For each line I and column J the associated parameters are $(i^l, k_1^l, k_2^l, k_3^l)$ for the lines and $(i^c, k_1^c, k_2^c, k_3^c)$ for the columns.
- An example of a derivative matrix is:

$$[\mathscr{D}^{12}]_{I,J} = \begin{cases} 1 & \text{if} \begin{cases} (i^l, k_1^l, k_2^l, k_3^l) = (i^c, k_1^c + 1, k_2^c + 1, k_3^c) \\ (i^l, k_1^l, k_2^l, k_3^l) = (i^c, k_1^c - 1, k_2^c - 1, k_3^c) \end{cases} \\ -1 & \text{if} \begin{cases} (i^l, k_1^l, k_2^l, k_3^l) = (i^c, k_1^c + 1, k_2^c - 1, k_3^c) \\ (i^l, k_1^l, k_2^l, k_3^l) = (i^c, k_1^c - 1, k_2^c + 1, k_3^c) \end{cases} \\ 0 & \text{otherwise} \end{cases}$$

Numerical Equation

- $\mathbf{M} \times \gamma \mathbf{B}$ is also expressed in matrix form $\mathrm{G}\mathbf{u}$.
- The projections onto the $\mathbf{e}_m$ axis are also defined by P^m.
- Calling $\mathbf{u}_0^z$ the corresponding vector for $\mathrm{P}^3\,\mathbf{u}(t=0)$, the Bloch-Torrey Equation is:

$$\mathscr{D}^t\mathbf{u} = \left(\mathrm{G} - \frac{\mathrm{P}^1+\mathrm{P}^2}{T_2} + \frac{\mathrm{P}^3}{T_1} + \left[\mathrm{D}_j{}^k{}_{,i}\mathrm{I} + \mathrm{D}_j{}^k\mathscr{D}^i - \Gamma^l{}_{ji}\mathrm{D}_l{}^k\mathscr{D}^k\right]\right)\mathbf{u} - \frac{\mathbf{u}_0^z}{T_1}$$

- Written in the form:

$$\mathscr{D}^t\mathbf{u} = \mathcal{S}\mathbf{u} + \mathbf{s}$$

Crank-Nicholson Method

- The Crank-Nicholson method is used:

$$\frac{\mathbf{u}^{+} - \mathbf{u}}{\Delta t} = \mathcal{S}\frac{\mathbf{u}^{+} + \mathbf{u}}{2} + \mathbf{s}$$

- The equation is reorganized:

$$\left(\mathrm{I} - \Delta t\,\frac{\mathcal{S}}{2}\right)\mathbf{u}^{+} = \left(\mathrm{I} + \Delta t\frac{\mathcal{S}}{2}\right)\mathbf{u} + \mathbf{s}$$

- Which is a simple linear system:

$$\mathcal{A}\mathbf{u}^{+} = \mathbf{b}$$

Algorithm

- build matrices $\mathscr{D}^i, \mathscr{D}^{ij}, \mathrm{P}^i$ and vector $\mathbf{u}_0^z$
- for all t
  - for all ξ
    - Build matrix G $(= \mathbf{M} \times \gamma\,(\mathbf{B} + \mathbf{x} \cdot \mathbf{G}))$
    - Multpily each lines with the coefficients : $\mathrm{D}_j{}^k, \Gamma^l{}_{ji}$
  - end
- Build matrix $\mathcal{A}$ and vector $\mathbf{b}$
- Solve for $\mathbf{u}^+$: $\mathcal{A}\,\mathbf{u}^+ = \mathbf{b}$
- end

Limitations and Future Work

- Size of the linear system:

$$N \simeq N_1 \simeq N_2 \simeq N_3 \simeq 64 \Rightarrow \text{size} = (3N^3)^2 \simeq 62.10^{10}$$

- Matrix $\mathcal{A}$ is sparse but not tridiagonal $\Rightarrow$ Time to invert the system.
  - Possibility of speeding-up by using the ADI (Alternative Direction Implicit) scheme ...