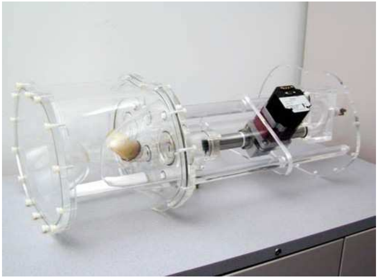
A Deformable Balloon for Tomography Motion Artifact Study

Damien Rohmer

November 21, 2006

Introduction of the Problem

- Mechanical system for deformation of a balloon
- Simulate simply the deformation of a heart
- Enable tomography measurement of the artifacts
- $$$



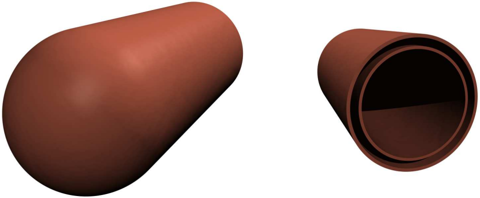
Goal

- Model the Equation of the deformation.
- Solve (numerically) to observe the behavior of the balloon.
- Perform a CT acquisition.

Model used

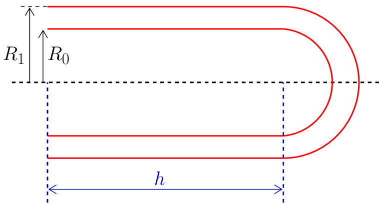
Table: Dimensions of the balloons

	h (cm)	R (cm)
interior	8.5	1.75
exterior	8.5	2.75



Approximations

- The problem is supposed to be planar (isotropic in the circumferential direction).
- The problem solved is static (no dynamic fluid mechanic).
- Gravity effect are neglected.



Notation of the curve

- One membrane is considered
- The 2D profil is parameterized with s.
- Every position on the profil is defined by the curve $\mathbf{c}$ such that

$$\mathbf{c}(s) = (c_x(s), c_y(s))$$

Pressure Action

- The action is constantly normal to the curve.
- The magnitude is constant (no gravity effect).
- The normal is called $\mathbf{n}(s)$ so $F_p(s) = f\,\mathbf{n}(s)$.
- The normal can be expressed with the curve:

$$\mathbf{F}_p(s) = f \frac{\begin{pmatrix} -c_y'(s) \\ c_x'(s) \end{pmatrix}}{\sqrt{{c_x'}^2(s) + {c_y'}^2(s)}}$$

- f can be expressed as the total force F divided by the area element

$$f = \frac{F^\alpha}{2\pi} \frac{1}{\int_{\mathbf{c}} c_y(u) \sqrt{{c_x'}^2(u) + {c_y'}^2(u)} \mathrm{d}u}$$

Reaction of the Membrane

- The elastic membrane tend to limit the deformations.
- Tend to reach the initial shape at rest.
- Can be expressed (linear approximation) in the case of a constant stiffness λ by

$$\mathbf{F}_e(s) = \lambda\,(\mathbf{c} - \mathbf{c}^0)''(s)$$

Addition of the damping Force

- A Damping force to decrease the Energy.
- Simulate by a fluid friction force.
- Does not change the final state.

$$\mathbf{F}_d = -\mu \frac{\partial \mathbf{c}}{\partial t}(s,t)$$

Static Equation

- Want to solve directly the final state.
- The Damping force is not used.
- There is no curve evolution through time

$$\mathbf{F}_p(\mathbf{c}) + \mathbf{F}_e(\mathbf{c}) = \mathbf{0}$$

Equation to solve

$$\left\{\begin{array}{l} -f\dfrac{c_y'(s)}{\sqrt{{c_x'}^2(s)+{c_y'}^2(s)}}+\lambda\left(c_x-c_x^0\right)''(s)=0 \\ f\dfrac{c_x'(s)}{\sqrt{{c_x'}^2(s)+{c_y'}^2(s)}}+\lambda\left(c_y-c_y^0\right)''(s)=0 \end{array}\right.$$

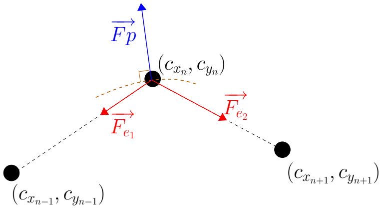
Or, calling $z = c_x + ic_y$ and $\mu = \frac{f}{\lambda}$, the equation is given in complex form

$$z'' + i\mu\frac{z'}{|z'|} = z^{0''}$$

Discretization

- The equation is spatially discretized to give the non linear system

$$\left\{ \begin{array}{l} \mathcal{F}\Big((c_{x_i}, c_{y_i})_{i \in [\![2, N-1]\!]} \Big) = \mathbf{0} \\ (c_{x_1}, c_{y_1}) = \mathbf{c}^0 \\ (c_{x_1}, c_{y_1}) = \mathbf{c}^N \end{array} \right.$$



Numerical Solution

- $\mathbf{Z}$ such that $\mathbf{Z}_{2n+1} = c_{x_n}$ and $\mathbf{Z}_{2n} = c_{y_n}$, the non linear system with $2N$ unknown is solved by **Newton's method**.

$$\begin{cases} \mathbf{Z}^{i+1} = \mathbf{Z}^{i} - \mathrm{D}\mathcal{F}^{-1}(\mathbf{Z}^{i})\,\mathcal{F}(\mathbf{Z}^{i}) \\ \\ \mathrm{D}\mathcal{F}_{ij}(\mathbf{Z}^{k}) = \dfrac{\partial \mathcal{F}_i}{\partial \mathbf{Z}^{i}_{j}}(\mathbf{Z}^{i}) \end{cases}$$

Problems

- The convergence is slow for large deformations.
- Oscillations spoil the stability during the iterations.
- The path between the initial and final step is not controled.

Second Method: Evolving Method

- **Idea:** To stay close from a physical solution during the iterations.
- **Method:** The curve is now evolving through time

$$\frac{\partial^2 \mathbf{c}}{\partial t^2}(s,t) = \mathbf{F}_p(\mathbf{c},s,t) + \mathbf{F}_e(\mathbf{c},s,t) + \mathbf{F}_d(\mathbf{c}_{,t},s,t)$$

New Equation

- A new equation has to be taken in account

$$
\begin{cases}
c_{x,tt} = & -\dfrac{F^{\alpha}}{2\pi} \dfrac{1}{\int_s c_y \sqrt{c_{x,s}^2 + c_{y,s}^2}\,\mathrm{d}s} \dfrac{c_{y,s}}{\sqrt{x_{x,s}^2 + c_{y,s}^2}} + \lambda \left(c_x - c_x^0\right)_{,ss} \\
& -\mu\, c_{x,t} \\
c_{y,tt} = & \dfrac{F^{\alpha}}{2\pi} \dfrac{1}{\int_s c_y \sqrt{c_{x,s}^2 + c_{y,s}^2}\,\mathrm{d}s} \dfrac{c_{x,s}}{\sqrt{x_{x,s}^2 + c_{y,s}^2}} + \lambda \left(c_y - c_y^0\right)_{,ss} \\
& -\mu\, c_{y,t}
\end{cases}
$$

- Looks not as good ...

Matrix Notation

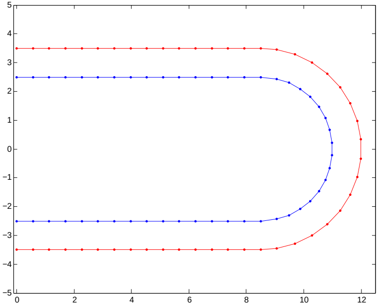
- The temporal order is decreased by the use of matrix

$$\mathbf{U} = \begin{pmatrix} \mathbf{c} \\ \mathbf{c}_{,t} \end{pmatrix} , \mathrm{M}_{sys} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} , \mathcal{F} = \begin{pmatrix} 0 \\ \sum_i \mathbf{F}_i \end{pmatrix}$$

$$\Rightarrow \mathbf{U}_{,t} = \mathrm{M}_{sys}\mathbf{U} + \mathcal{F}$$

Discretization step

- The system is discretized in the spatial domaine: PDE⇒ODE in time (method of lines)
- **U** is a vector of $2N$ unknown. M_{sys} is a $2N \times 2N$ identity block matrix.



Method of solution

- For stability reasons, the parabolic equation equation is solve by an implicit method:

$$\mathbf{U}(t+\Delta t) = \mathbf{U}(t) + \Delta t\left(\mathrm{M}_{sys}\mathbf{U}(t+\Delta t) + \mathcal{F}(t+\Delta t)\right)$$

- Problem: the new step of the Force is unknow and non-linear
- Need a linearization.

Elastic Term

- Already linear

$$\begin{pmatrix} 0 \\ \mathbf{F}_e \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ \lambda\delta^2 & 0 \end{pmatrix} \mathbf{U} - \lambda\delta^2 \begin{pmatrix} 0 \\ \mathbf{c}^0 \end{pmatrix} ,$$

where δ^2 is the discrete operator of the second derivative.

Damping term

- Linear too;

$$\begin{pmatrix} 0 \\ \mathbf{F}_d \end{pmatrix} = -\mu \begin{pmatrix} 0 & 0 \\ 0 & \mathrm{I} \end{pmatrix} \mathbf{U}$$

Pressure Action

- Need to be linearized.
- Use of multivariable Taylor expansion:

$$\mathbf{F}_p(t+\Delta t) \simeq \mathbf{F}_p(t) + \Delta t \sum_j \frac{\partial \mathbf{F}_p}{\partial \mathbf{c}^j} \mathbf{c}^j_{,t}(t+\Delta t)$$

$$\Rightarrow \begin{pmatrix} 0 \\ \mathbf{F}_p(t+\Delta t) \end{pmatrix} = \begin{pmatrix} 0 \\ \mathbf{F}_p(t) \end{pmatrix} + \Delta t \begin{pmatrix} 0 & 0 \\ 0 & \frac{\partial \mathbf{F}}{\partial \mathbf{c}} \end{pmatrix} \mathbf{U}$$

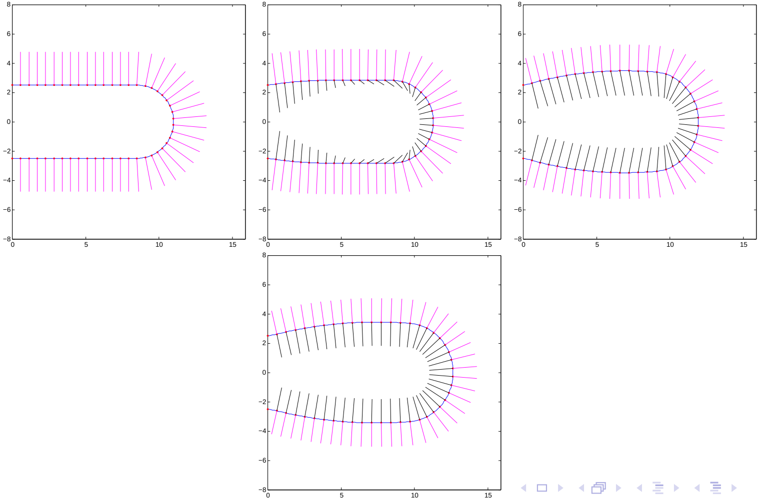
Iterative Solution

- The numerical method is defined by the solution of the block matrix equation

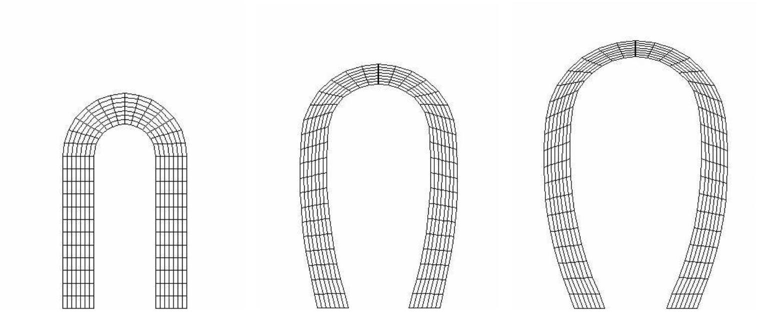
$$\begin{pmatrix} \mathrm{I} & -\Delta t\,\mathrm{I} \\ -\Delta t\lambda\delta^2 & \mathrm{I}-(\Delta t)^2\dfrac{\partial \mathbf{F}_p}{\partial \mathbf{c}} \end{pmatrix} \begin{pmatrix} \mathbf{c} \\ \mathbf{c}_{,t} \end{pmatrix}^{i+1}$$

$$= \begin{pmatrix} \mathrm{I} & 0 \\ 0 & (1-\Delta t\,\mu)\mathrm{I} \end{pmatrix} \begin{pmatrix} \mathbf{c} \\ \mathbf{c}_{,t} \end{pmatrix}^{i} + \begin{pmatrix} 0 \\ \mathbf{F}_p(t) \end{pmatrix} - \begin{pmatrix} 0 \\ \mathbf{F}_d^0 \end{pmatrix}$$

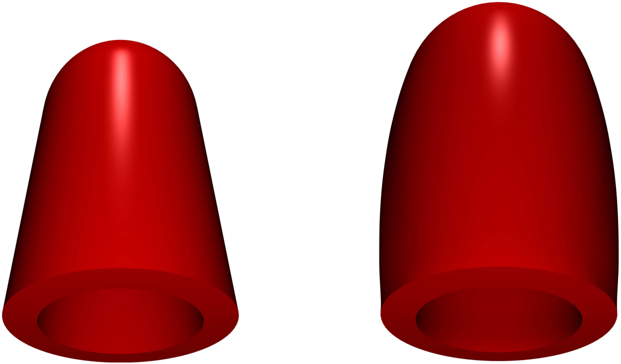
Evolution of the Shape and Forces



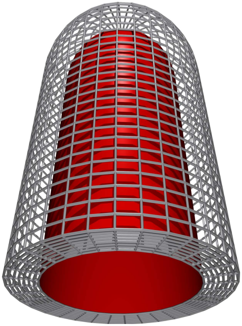
Mesh Deformation

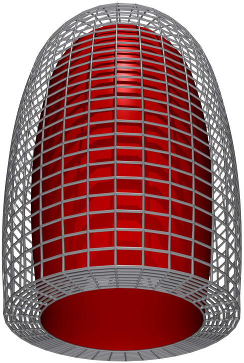


3D Visualization



3D Mesh Visualization





Artifacts Simulation

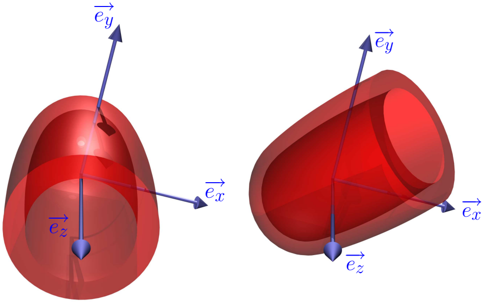
1. A Fast Acquisition (CT)
2. A Slow Acquisition (PET/SPECT: but no noise...)

Assumptions

- Dynamic of the balloon model the motion of the ventricle.
- Projections free of attenuation.
- Projections free of noise.
- Camera rotates with a perfect circle centered around the heart
- Projections performed on a voxelized volume where is voxel has the same concentration of tracer.

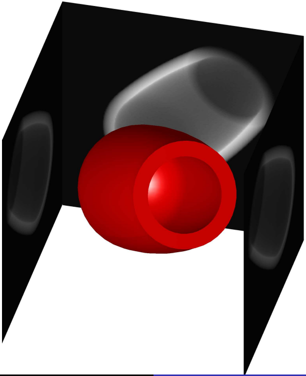
Method

- Rotation of the heart to 45° as in the torso.
- Ejection Fraction set to 60%
- Assume 65 beats per minutes.

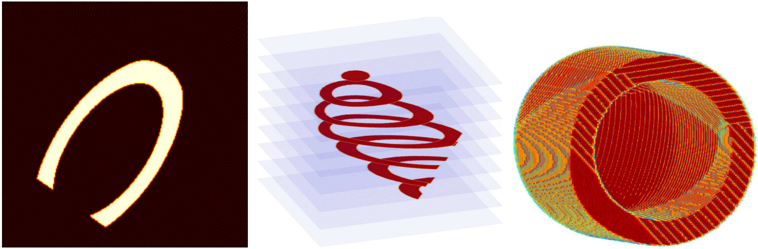


Projections

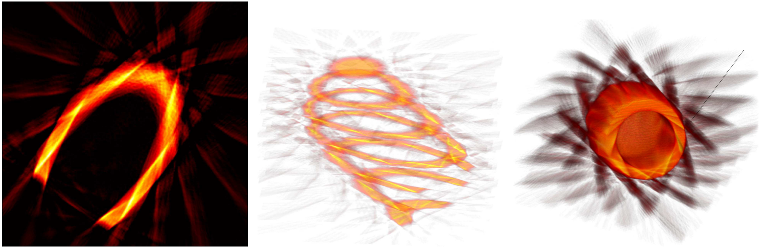
- **Fast case:** Rotation of 2° per second for the camera (instantaneous projections).
- **Slow Case:** Long enough (> 1 beat).



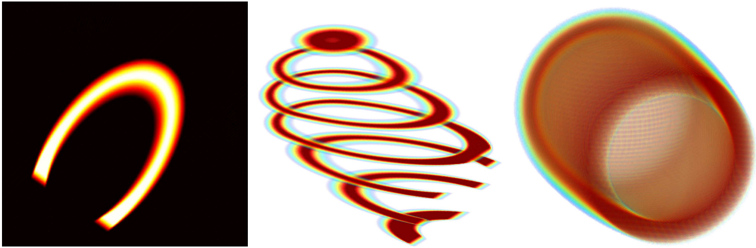
Reconstruction: Perfect Case



Reconstruction: Fast Aquisition



Reconstruction: Slow Aquisition



Conclusion

- Two methods of solutions.
- Same results, but the PDE is much more stable.
- Gives realistic deformations for a plastic balloon.
- Enables CT simulations.

Limitations:

- Extremely simplified model of heart/platic balloon.
- Need to be validated by a CT acquisition.